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An explicit construction of ruled surfaces[☆]

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ABSTRACT

The main goal of this paper is to give a general algorithm to compute, via computer-algebra systems, an explicit set of generators of the ideals of the projective embeddings of ruled surfaces, i.e. projectivizations of rank two vector bundles over curves, such that the fibers are embedded as smooth rational curves.

There are two different applications of our algorithm. Firstly, given a very ample linear system on an abstract ruled surface, our algorithm allows computing the ideal of the embedded surface, all the syzygies, and all the algebraic invariants which are computable from its ideal as, for instance, the k-regularity. Secondly, it is possible to prove the existence of new embeddings of ruled surfaces.

The method can be implemented over any computer-algebra system able to deal with commutative algebra and Gröbner-basis computations. An implementation of our algorithms for the computer-algebra system `Macaulay2` (cf. [Daniel R. Grayson, Michael E. Stillman, Macaulay 2, a software system for research in algebraic geometry, 1993. Available at http://www.math.uiuc.edu/Macaulay2/]) and explicit examples are enclosed.

0. Introduction and notation

Let E be a rank 2 vector bundle over a smooth, genus q, curve C. It is known that any such vector bundle E, regarded as a sheaf, is an extension of invertible sheaves. If E is a normalized vector bundle, i.e. $H^0(C, E) \neq 0$ but $H^0(C, E \otimes G) = 0$ for any line bundle G of negative degree, then E fits into a short exact sequence

$$0 \to \mathcal{O}_C \to E \to L \to 0, \tag{0.1}$$

and $L = \det(E)$.

We consider the geometrically ruled surface $X := \mathbb{P}(E)$, endowed with the natural projection $p : \mathbb{P}(E) \to C$. In this case $Pic(X) \cong \mathbb{Z} \oplus p^*Pic(C)$, where $\mathbb{Z}$ is generated by the tautological divisor of X, i.e. a divisor C_0, image of a section $\sigma_0 : C \to X$ with minimal self-intersection. According to this notation, every divisor on X is linearly (resp. numerically) equivalent to $aC_0 + p^*B$ where B is a degree b divisor of C (resp. $aC_0 + bf$, where f is the numerical class of a fiber of p).

We choose a very ample divisor A on X and we consider the polarized ruled surface (X, A), i.e., X embedded in $\mathbb{P}^{h^0(X,A)-1}$ by $|A|$: we aim to give an algorithm to compute a set of generators of its ideal I_X in the ring $S(V) := \bigoplus_{i\geq 0} S^i(V)$, the symmetric algebra of $V = H^0(X, A)$. The algorithm requires the knowledge of the following data: a set of generators of the ideal I_C of any embedded image of C in some projective space; the (Weil) divisor B of C; and the extension class giving

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Notation Table

$\mathbb{K}$	base field, usually $\mathbb{C}$ (but also finite fields are considered)
$\mathbb{P}^n$	projective n-dimensional space over $\mathbb{K}$
$\mathbb{P}(E)$	projectivization of the rank 2 vector bundle E over a smooth curve C, C_0 is its tautological divisor, $p : \mathbb{P}(E) \to C$ the natural projection, and f the numerical class of a generic fiber of p
$c_i(E)$	ith Chern class of E
$\mathbf{F}_{e,q}$	ruled surface of invariant $e := -\deg[c_1(E)] \geq -q$ over a smooth, genus q, curve C
$\equiv$	numerical equivalence
$*$	means duality
$\lvert D\rvert$	linear system of effective divisors linearly equivalent to the divisor D
$I_W (\mathcal{I}_W)$	ideal (ideal sheaf) of a projective variety $W \subset \mathbb{P}^n$
K_W	canonical divisor of a smooth variety W
$g(W)$	sectional genus of a smooth variety $W \subset \mathbb{P}^n$
$H^0_*(W, \mathcal{F})$	$\bigoplus_{t\geq 0} H^0(W, \mathcal{F} \otimes \mathcal{O}_W(t))$ for any sheaf $\mathcal{F}$ on $W \subset \mathbb{P}^n$
$\widetilde{M}$	sheaf of $\mathcal{O}_W$-modules associated to any S-module M, where S is the coordinate ring of a smooth variety W
$S(V)$	$\oplus_{n\geq 0} S^n(V)$ symmetric algebra of the vector space V
$S(E)$	$\oplus_{n\geq 0} S^n(E)$ symmetric $\mathcal{O}_W$-algebra of the vector bundle E over a variety W
$\mu(E)$	$\deg E/\operatorname{rk} E$, slope of the vector bundle E
$\mu^-(E)$	$\min\{\mu(Q)\mid E \to Q \to 0\}$

E (in turn, we will need to give a specific morphism between two modules corresponding to the extension class given by Eq. (0.1)).

Ampleness conditions for the divisor A are classical and well known (cf. e.g. [14]). In particular, by Nakai's criterion, denoting with $e := -\deg E$ the invariant of X, an ample divisor A is numerically equivalent to $aC_0 + bf$ with $a \geq 1$ and $b > ae$ if $e \geq 0$ or $b > ae/2$ if $e < 0$. On the contrary the very ampleness condition for A has to be checked case by case with some criteria, e.g. Reider's criterion (cf. [20]) or by looking at the image of X by $|A|$.

Let $\mathbb{K}$ be the base field. Our main result is the following theorem, the algorithm being included in its proof:

Main Theorem (*char* $\mathbb{K} = 0$ *or* $q \leq 1$ *for* $k > 1$). *Let $C \subset \mathbb{P}^m$ be a smooth curve C of genus q, B a divisor on C and L a line bundle over C. Consider a normalized rank 2 vector bundle $E \in Ext^1(L, \mathcal{O}_C)$ over C given by an extension $0 \to \mathcal{O}_C \to E \to L \to 0$ and suppose that, for some positive integer k, the divisor $A = kC_0 + p^*B$ on the surface $X = \mathbb{P}(E)$ is very ample.*

Then there is an algorithm yielding a set of generators of the ideal I_X of the embedded X in $\mathbb{P}^{h^0(X,A)-1} = \mathbb{P}(H^0(X, A)^*)$ by $|A|$.

A few words about the algorithm. We already listed at the beginning the data required by the algorithm. To be more precise, the algebraic data needed by the algorithm are:

(1) the ideal I_C of the embedded curve $C \subset \mathbb{P}^m$;
(2) three divisors on C: B, an auxiliary effective divisor D, satisfying some technical conditions (see (1.3) and (3.1)), such that $D - B$ is effective and $|L \otimes \mathcal{O}_C(D)| \neq \emptyset$, and another auxiliary divisor $D' \in |L \otimes \mathcal{O}_C(D)|$;
(3) an element in $Ext^1_S(I^*_{D'}, I^*_D)$.

Let us see how these algebraic data are related to the data at the beginning. Let S be the coordinate ring of C. The first technical condition for D, i.e., (1.3), will guarantee that the graded S-module $H^0_*(C, E \otimes \mathcal{O}_C(D)) = \oplus_{i\geq 0} H^0(C, E \otimes \mathcal{O}_C(D) \otimes \mathcal{O}_C(i))$ is an extension in $Ext^1_S(H^0_*(C, L \otimes \mathcal{O}_C(D)), H^0_*(C, \mathcal{O}_C(D)))$. We will moreover see that $H^0_*(C, L \otimes \mathcal{O}_C(D)) = I^*_{D'}$ and $H^0_*(C, \mathcal{O}_C(D)) = I^*_D$. Therefore, there is a 1:1 correspondence between the elements in $Ext^1_S(I^*_{D'}, I^*_D)$ as in (3) and the vector bundles $E \in Ext^1(L, \mathcal{O}_C)$.

The technical conditions for D will be explicitly given and we will see that it is always possible to determine a minimal degree for D such that they will hold. Notice also that the polarized surface (X, A) is independent of the choices of the projective model for C, of D, and of $D' \in |L - D|$.

We also remark that two assumptions in the main theorem can be replaced by computational checks: the assumption char $\mathbb{K} = 0$ (if $q > 1$) and the very ampleness assumption for $A = kC_0 + p^*B$.

The restriction on the characteristic in the statement of the main theorem for $k \geq 2$ is required only to guarantee the k-normality of an isomorphic image of X involved in the algorithm and depending on the choice of D. In finite characteristic the algorithm is still valid modulo performing a computational check of the k-normality and eventually changing the choice of D (see Remark 3.6).

Even without the very ampleness assumption for A, the algorithm computes the ideal of the image of X by the rational map $\phi_{|A|}$ associated to $|A|$. A computational check that $\phi_{|A|}(X)$ is a smooth surface of degree A^2 and that $|A|$ has empty base locus ensures then that $\phi_{|A|}$ is an embedding (see Remarks 3.7 and 3.8).

The algorithm is straightforward in the case of scrolls, when $k = 1$. Instead, in the case $k \geq 2$ it firstly computes a suitable scroll and then its image by a rational map associated to a suitable linear system. Hence it is useful to divide the main theorem into two steps. In Section 1 we will give and prove Theorem A, concerning the case of scrolls. In Section 3 we will give and prove Theorem B, concerning the cases $k \geq 2$. The main theorem will follow from Theorems A and B.

In Sections 2 and 4 we will give applications of the algorithm to study some properties of surfaces whose existence is known. In Section 2 we describe a family of scrolls of degree 8 with sectional genus 2 (cf. [7]) and a family of scrolls of degree 6 with sectional genus 1, while in Section 4 we describe conic bundles of degree 8 with sectional genus 3, cubic bundles of degree 9 with sectional genus 4 and a ruled surface having cubic fibers.

In Section 5 we describe some varieties related to ruled surfaces with conic fibers.

In Section 6 we use our algorithm to construct some embedded surfaces whose existence was not known. To do this, we will consider a large set of ruled surfaces X treated in [18, Section 3]. There the author proved the very ampleness of some divisors $\mathcal{L}_0 \equiv aC_0 + bf$ by applying Reider's criterion in a convenient way, but she was unable to decide whether also the subsystems given by $\mathcal{L} = \mathcal{L}_0 - \sum_1^t p_i$ are very ample for low values of t and generic choices of the points p_i's. This amounts to asking whether some projections of $(X, \mathcal{L}_0)$ are smooth. By our algorithm we will be able to get a set of generators for the ideal of $(X, \mathcal{L}_0)$, to project $(X, \mathcal{L}_0)$ from some random points on it, and then to test directly whether such projections are smooth.

In this way we prove that there exist geometrically ruled surfaces X, over genus q curves, of invariant e, such that the linear systems $|\mathcal{L}|$ in the following table are very ample. X will be embedded by $|\mathcal{L}|$ as a surface of degree d and sectional genus $g = 7$.

	g	d	$h^0(\mathcal{L})$	q	$\lvert\mathcal{L}\rvert$	e	t
(9)	7	$18 - t$	$12 - t$	1	$3C_0 + 3f - \sum_1^t p_i$	0	$1 \leq t \leq 5$
(10)	7	$16 - t$	$10 - t$	1	$4C_0 - \sum_1^t p_i$	-1	$2 \leq t \leq 3$
(11)	7	$15 - t$	$9 - t$	1	$5C_0 - \sum_1^t p_i$	-1	$1 \leq t \leq 2$
(13)	7	$16 - t$	$9 - t$	2	$2C_0 + 2f - \sum_1^t p_i$	-2	$t = 2$
(13′)	7	$16 - t$	$9 - t$	2	$2C_0 + 3f - \sum_1^t p_i$	-1	$1 \leq t \leq 2$

There are other methods to get explicit equations of embedded ruled and (non-ruled) surfaces, but they work only for low codimension and they do not allow having some control over the "geometry" of the polarized surface (the curve C, the extension giving E, and the divisor B).

Indeed, if X has codimension 2, i.e. $X \subset \mathbb{P}^4$, then there exist two sheaves $\mathcal{F}$ and $\mathcal{G}$ with $\operatorname{rk} \mathcal{G} = \operatorname{rk} F + 1$ and a map $\Phi : \mathcal{F} \to \mathcal{G}$ such that the Eagon–Northcott complex defined by the maximal minors of Φ identifies coker Φ with the ideal sheaf of X. The sheaves $\mathcal{F}$ and $\mathcal{G}$ are then constructed starting from the cohomology table of $\mathcal{I}_X$. This constructing method was introduced in [8] and it is largely used to construct surfaces in $\mathbb{P}^4$ (c.f. also [9] for a further description and a nearly up-to-date list of references). If X has codimension 3, this type of construction can be still performed using the Pfaffian complex instead of the Eagon–Northcott complex: if X is a codimension 3 subcanonical scheme in $\mathbb{P}^5$ a locally free resolution of its ideal sheaf is still known (c.f. [22]).

A final remark. The algorithm requires the knowledge of the embedded curve C which is the base of the ruling of the surface. If instead, in an example, we do not have a projective model of C there is the further mathematical problem to construct such a model, at least for a random curve C (random in its moduli space). There is no general method to determine embedded curves at random, but it is a classical topic how to parametrize curves of low genus ($g \leq 10$) by using nodal plane models, and an explicit method to parametrize smooth space curves up to genus 14 is known and illustrated in [21].

We use the computer-algebra program `Macaulay2` [11] to implement the algorithms and the examples.

1. Construction of scrolls

In this section we develop an algorithm to compute a set of generators of the ideal of embedded scroll surfaces:

Theorem A. *Let $C \subset \mathbb{P}^m$ be a smooth curve C of genus q, B be a divisor on C and L a line bundle over C. Consider a normalized rank 2 vector bundle $E \in \mathrm{Ext}^1(L, \mathcal{O}_C)$ over C given by an extension $0 \to \mathcal{O}_C \to E \to L \to 0$ and suppose that the divisor $A = C_0 + p^*B$ on the surface $X = \mathbb{P}(E)$ is very ample, where $p : \mathbb{P}(E) \to C$ is the natural projection.*

Then there is an algorithm yielding a set of generators of the ideal I_X of the embedded X in $\mathbb{P}^{h^0(X,A)-1} = \mathbb{P}(H^0(X, A)^*)$ by $|A|$.

A description of the explicit form of the data required by the algorithm is given in the introduction, immediately after the statement of the main theorem.

At first, let us point out the basic idea of the method. Let I_C be the ideal of the curve C in $\mathbb{P}^m = \mathrm{Proj}(R)$, where $R = \mathbb{K}[x_0, .., x_m]$, and let $S := R/I_C$ be the coordinate ring of $C \subset \mathbb{P}^m$. Our strategy will be to get a presentation of the S-module M defined as

$$M := H^0_*(C, E \otimes \mathcal{O}_C(B)) = \oplus_{i \geq 0} H^0(C, E \otimes \mathcal{O}_C(B + iH)),$$

where H is a hyperplane divisor of $C \subset \mathbb{P}^m$. Then we will apply a straightforward computation to get a set of generators of I_X. Details will be given later in the proof of the theorem. The short exact sequence ((0.1) implies the exactness of

$$0 \to \mathcal{O}_C(B) \to E \otimes \mathcal{O}_C(B) \to L \otimes \mathcal{O}_C(B) \to 0, \tag{1.1}$$

from which we will derive the desired presentation of M.

Now, we give here some lemmas needed for the proof of the theorem.

Lemma 1.1. *Let I_C be the ideal of a smooth curve C in a projective space $\mathbb{P}^m$ and let S be the coordinate ring of C. Let D be an effective divisor on C. Then the S-modules $H^0_*(C, \mathcal{O}_C(D))$ and $(I_D)^* := Hom_S(I_D, S)$ are naturally isomorphic as (graded) S-modules, where $I_D \subset S$ is the ideal of the divisor D.*

Proof. Let us recall the following well-known result on local cohomology (cf. [10, Thm. A4.1]). Let S be a graded noetherian ring with degree 0 part a field, $\mathfrak{m}$ the maximal ideal generated by the degree 1 part of S and M a finitely generated graded S-module. Then there is a natural exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \oplus_{i\geq 0} H^0(\mathrm{Proj}\, S, \widetilde{M}(i)) \to H^1_{\mathfrak{m}}(M) \to 0,$$

where $H^i_{\mathfrak{m}}(M)$ denotes the ith local cohomology group of M with respect to $\mathfrak{m}$. By using this sequence, whenever $W \subset \mathbb{P}^m$ is a variety, S is its coordinate ring, and M is any graded S-module, one obtains that $M \cong H^0_*(W, \widetilde{M})$ if $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$. Moreover a sufficient condition for obtaining $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$ is that $depth(\mathfrak{m}, M) \geq 2$ by [14, Ch. III, Ex. 3.4 and 3.3]. For more details and related results, see [13] (in particular Prop. 2.2 and Thm. 3.8).

In our case $W = C$, $S := \mathbb{K}[x_0, \ldots, x_m]/I_C$ is the coordinate ring of the curve C in $\mathbb{P}^m$, $\mathfrak{m} := (x_0, \ldots, x_m)$ is the image of the irrelevant ideal of $\mathbb{K}[x_0, \ldots, x_m]$ in S, and $M = (I_D)^*$. Recall that S is the coordinate ring of an affine cone over a curve, and therefore I_D, as well as $(I_D)^*$, is not necessarily a projective S-module, since it may be not locally free if we localize it at the vertex of the cone. Therefore we proceed as follows.

Firstly, notice that $\mathrm{depth}(\mathfrak{m}, S) = 2$ and that a regular sequence for S is also a regular sequence for I_D, since I_D is a submodule of S, thus $\mathrm{depth}(\mathfrak{m}, I_D) \geq \mathrm{depth}(\mathfrak{m}, S) = 2$ and $I_D = H^0_*(C, \mathcal{O}_C(-D))$.

Secondly, we claim that, if $t_1, \ldots, t_d \in \mathfrak{m}$ is a regular sequence for S, then it is also a regular sequence for $(I_D)^* = Hom(I_D, S)$. We argue by contradiction. Suppose that t_i is a 0-divisor for $(I_D)^* \mathrm{mod}(t_1, \ldots, t_{i-1})$. Then there exists a non-zero morphism $\varphi \in (I_D)^* \mathrm{mod}(t_1, \ldots, t_{i-1})$ s.t. $t_i\varphi = 0 \,\mathrm{mod}(t_1, \ldots, t_{i-1})$. Take an $x \in I_D$ s.t. $\varphi(x) \neq 0$ in $S/(t_1, \ldots, t_{i-1})$: from $t_i\varphi(x) = 0$ in $S/(t_1, \ldots, t_{i-1})$ we get that t_i is a 0-divisor in $S/(t_1, \ldots, t_{i-1})$, a contradiction. Moreover $t_1, \ldots, t_d \in \mathfrak{m}$, hence we have $(t_1, \ldots, t_d)(I_D)^* \neq (I_D)^*$. Indeed, if this is not the case, then $\mathfrak{m}(I_D)^* = (I_D)^*$ and therefore there exists an element $r \in \mathfrak{m}$ such that $(1 - r)(I_D)^* = 0$, cf. [10, Cor. 4.7]. In particular, considering the inclusion $\iota : I_D \to S$, we have $(1 - r)\iota = 0$ and therefore $1 - r$ is a 0-divisor in S. Since S is an integral domain, it follows that $r = 1$, which is absurd since $r \in \mathfrak{m}$. We conclude that $\mathrm{depth}(\mathfrak{m}, (I_D)^*) \geq \mathrm{depth}(\mathfrak{m}, S) = 2$ and therefore we get $(I_D)^* = H^0_*(C, \widetilde{[(I_D)^*]}) = H^0_*(C, \mathcal{O}_C(D))$. □

Lemma 1.2. *Let S be a commutative ring, F and G be two S-modules with free resolutions:*

$$F^\bullet : \quad \cdots \to F_3 \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi} F \to 0,$$

$$G^\bullet : \quad \cdots \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0 \xrightarrow{\psi} G \to 0.$$

Then any morphism $\varphi \in Hom_S(F_1, G_0)$ satisfying $\psi \circ \varphi \circ \phi_2 = 0$, i.e. inducing a morphism in $Hom_S(\ker\phi, G) = Hom_S(\mathrm{im}\,\phi_1, \mathrm{coker}\,\psi_1)$, determines an extension $M \in Ext^1_S(F, G)$ and, conversely, any extension is determined by such a morphism.

Moreover, the module $M \in Ext^1_S(F, G)$ corresponding to φ has presentation

$$\begin{pmatrix} \phi_1 & 0 \\ \varphi & \psi_1 \end{pmatrix} : F_1 \oplus G_1 \to F_0 \oplus G_0.$$

Proof. The proof is standard. For completeness, since this proof is important to implement our algorithm, we shortly repeat it here (for some references cf. [12, pag. 722] or [10, Ex. A3.26]). Consider the module $K = \ker\phi$. The short exact sequence $0 \to K \to F_0 \xrightarrow{\phi} F \to 0$ induces by duality

$$Hom_S(F_0, G) \to Hom_S(K, G) \to Ext^1_S(F, G) \to Ext^1_S(F_0, G) = 0.$$

Therefore $Ext^1_S(F, G) \cong Hom_S(K, G)/Hom_S(F_0, G)$.

In the same way the short exact sequence $0 \to \ker\phi_1 \to F_1 \xrightarrow{\phi_1} K \to 0$ gives $0 \to Hom_S(K, G) \to Hom_S(F_1, G) \to Hom_S(\ker\phi_1, G)$. Hence $Hom_S(K, G)$ is the kernel of the second map and we can identify $Hom_S(K, G)$ with the set of morphisms $\eta \in Hom_S(F_1, G)$ whose restriction to $\ker\phi_1$ is zero or equivalently, since $\ker\phi_1 = \mathrm{im}\,\phi_2$, whose composition $\eta \circ \phi_2$ is zero. Since F_1 is projective, the surjectivity of $G_0 \to G$ gives the surjectivity of $Hom_S(F_1, G_0) \to Hom_S(F_1, G)$.

Conversely, a morphism $\varphi \in Hom_S(F_1, G_0)$ satisfying the hypothesis determines by composition a morphism $\eta \in Hom_S(F_1, G)$ satisfying $\eta \circ \phi_2 = 0$ which therefore is an element of $Hom_S(K, G)$: its equivalence class in $Hom_S(K, G)/Hom_S(F_0, G)$ determines an extension $M \in Ext^1_S(F, G)$, as desired.

To compute a presentation of such an extension M, let us denote with ι the inclusion $K \to F_0$, and with $\varphi' \in Hom_S(K, G)$ the morphism induced by φ. Then the module M is the quotient $(F_0 \oplus G)/\operatorname{im}(\iota \oplus \varphi')$, which is the cokernel of the morphism $\begin{pmatrix} \phi_1 & 0 \\ \varphi & \psi_1 \end{pmatrix} : F_1 \oplus G_1 \to F_0 \oplus G_0$. □

Lemma 1.3. *Let $W \subset \mathbb{P}^m$ be a smooth algebraic variety and let $\mathcal{E}$ be a locally free sheaf on W. Denote by $S := \mathbb{K}[x_0, \dots, x_m]/I_W$ the coordinate ring of W in $\mathbb{P}^m$. Suppose further that the tautological bundle $\tau_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of $\mathbb{P}(\mathcal{E})$ is very ample. Then, given a presentation of the S-module $M := H^0_*(W, \mathcal{E})$, there is an algorithm yielding a set of generators of the ideal $I_{\mathbb{P}(\mathcal{E})}$ of the embedded variety $\mathbb{P}(\mathcal{E})$ by the complete linear system $|\tau_{\mathbb{P}(\mathcal{E})}|$.*

Proof. Notice that $H^0(\mathbb{P}(\mathcal{E}), \tau_{\mathbb{P}(\mathcal{E})}) = H^0(W, \mathcal{E})$ and let $h^0(W, \mathcal{E}) = n + 1$. The embedding ι associated to $|\tau_{\mathbb{P}(\mathcal{E})}|$ comes with a map of sheaves of rings $\iota^\# : \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, induced by mapping $n + 1$ new variables $y_0, \dots, y_n$ to a basis of $H^0(W, \mathcal{E}) = H^0(\mathbb{P}(\mathcal{E}), \tau_{\mathbb{P}(\mathcal{E})})$. The ideal sheaf $\widetilde{I}_{\mathbb{P}(\mathcal{E})}$ is the kernel of this map.

Let $M' \subset M$ be the S-submodule generated by a basis of $H^0(W, \mathcal{E})$. If ϕ is the given free presentation of M, we can compute a free presentation ϕ' of M':

$$M_1 \xrightarrow{\phi'} M_0 \to M' \to 0,$$

where $\operatorname{rk} M_0 = n + 1$ and the generators of M_0 map to the chosen basis of $H^0(W, \mathcal{E})$.

Let us consider in $S[y_0, \dots, y_n]$ the ideal I given by

$$I := \begin{pmatrix} y_0 & \dots & y_n \end{pmatrix} \cdot \phi', \tag{1.2}$$

where, by abuse of language, we use again ϕ' to denote the matrix associated to the map ϕ'. The ideal $I_{\mathbb{P}(\mathcal{E})}$ is given by the polynomial relations among the $\{y_0, \dots, y_n\}$ in the saturation of I with respect to the ideal $(x_0, \dots, x_m) \subset S[y_0, \dots, y_n]$. Therefore, $I_{\mathbb{P}(\mathcal{E})}$ can be obtained by saturating I with respect to the ideal $(x_0, \dots, x_m)$ and intersecting this new ideal with the subring $\mathbb{K}[y_0, \dots, y_n]$. □

Remark 1.4. Eq. (1.2) yields a presentation, as a $S[y_0, \dots, y_n]$-module, of the S-algebra generated by $H^0(W, \mathcal{E})$ in $S(H^0_*(W, \mathcal{E}))$. If M is generated by $H^0(W, \mathcal{E})$, then M admits a presentation $\oplus_{j=0}^s S(-l_j) \xrightarrow{\phi} \oplus_{i=0}^n S \to M \to 0$ and the S-algebra $S(H^0_*(W, \mathcal{E}))$ has a presentation

$$\oplus_{j=0}^s S[y_0, \dots, y_n](-l_j) \xrightarrow{(\dots, \sum_{i=0}^n y_i \phi_{ij}, \dots)} S[y_0, \dots, y_n] \to S(H^0_*(W, \mathcal{E})) \to 0.$$

If this is not the case, i.e. M is not generated by the minimal degree part, let $\oplus_{j=0}^s S(-l_j) \xrightarrow{\phi} \oplus_{i=0}^n S(-h_i) \to M \to 0$ be a presentation of M. Then the S-algebra $S(H^0_*(W, \mathcal{E}))$ has still a presentation as above, but now the ring $S[y_0, \dots, y_n]$ is weighted, y_i having weight h_i.

Remark 1.5 (char $\mathbb{K} = 0$ *or* $q \le 1$). Let us assume that $W = C \subset \mathbb{P}^m$ is a smooth projective curve of genus q and $\mathcal{E}$ is a vector bundle over C. If $\mu^-(\mathcal{E}) \ge 2q$ and $\deg(\mathcal{O}_C(1)) \ge 2q$ and moreover one of these two inequalities is strict, then $H^0(C, \mathcal{E})$ generates $H^0_*(C, \mathcal{E})$ as S-module, where S denotes the coordinate ring of C in $\mathbb{P}^m$.

Proof. This is a direct application of Theorem 2.1 of [6]: under these assumptions the map $H^0(C, E) \otimes H^0(C, \mathcal{O}_C(t)) \to H^0(C, E \otimes \mathcal{O}_C(t))$ is surjective $\forall t \ge 0$. □

Proof of Theorem A. Let S be the coordinate ring of C in $\mathbb{P}^m$ and let H be the divisor induced on C by a hyperplane section of $\mathbb{P}^m$. Recall the assigned exact sequence in the statement: $0 \to \mathcal{O}_C \to E \to L \to 0$ and let D be any effective divisor on C such that $D - B$ is effective, $|L \otimes \mathcal{O}_C(D)| \ne \emptyset$, and

$$H^1(C, \mathcal{O}_C(D + jH)) = 0 \quad \forall j \ge 0. \tag{1.3}$$

Note that by choosing the degree of D big enough, it is always possible to find such a divisor D.

By condition (1.3)) there is a short exact sequence of S-modules

$$0 \to H^0_*(C, \mathcal{O}_C(D)) \to H^0_*(C, E \otimes \mathcal{O}_C(D)) \to H^0_*(C, L \otimes \mathcal{O}_C(D)) \to 0,$$

implying that $H^0_*(C, E \otimes \mathcal{O}_C(D))$ can be obtained as an extension in $Ext^1_S(H^0_*(C, L \otimes \mathcal{O}_C(D)), H^0_*(C, \mathcal{O}_C(D)))$.

Since $|L \otimes \mathcal{O}_C(D)| \ne \emptyset$, there exists an effective divisor $D' \in |L \otimes \mathcal{O}_C(D)|$. Applying Lemma 1.1 to the divisors D and D', we easily get explicit free resolutions of the S-modules $H^0_*(C, \mathcal{O}_C(D)) = I^*_D$ and $H^0_*(C, L \otimes \mathcal{O}_C(D)) = I^*_{D'}$.

Given the two resolutions of I_D^* and of $I_{D'}^*$, since $H^0_*(C, E \otimes \mathcal{O}_C(D))$ is an extension in $Ext^1_S(I^*_{D'}, I^*_D)$, we can apply Lemma 1.2 to get a presentation of $H^0_*(C, E \otimes \mathcal{O}_C(D))$. The tensorization of $H^0_*(C, E \otimes \mathcal{O}_C(D))$ with $I_{D-B} = H^0_*(C, \mathcal{O}_C(-D+B))$ yields an S-module M' whose associated coherent sheaf is $E \otimes B$.

With a computer-algebra system we compute a presentation of the module of the global sections of the coherent sheaf associated to M': this is precisely the module $M := H^0_*(C, E \otimes \mathcal{O}_C(B)) = H^0_*(C, \widetilde{M}')$.

Finally, from a presentation of M, Lemma 1.3 explains how to get a set of generators of I_X, where $X = \mathbb{P}(E)$ is embedded by the very ample divisor $A = C_0 + p^*B$ (the tautological divisor of $\mathbb{P}(E \otimes \mathcal{O}_C(B))$).

The algorithm based on the above considerations is therefore the following:

(1) choose on $C \subset \mathbb{P}^m$ an effective divisor D satisfying (1.3) and such that $D - B$ is effective, $|L \otimes \mathcal{O}_C(D)| \neq \emptyset$;
(2) choose a divisor $D' \in |L \otimes \mathcal{O}_C(D)|$ and, from the ideal of C and from L, compute S and a set of generators of the ideals I_D and $I_{D'}$ (as S-modules);
(3) identify the extension in $Ext^1_S(I_{D'}, I_D)$ corresponding to the given extension of E as an element in $Ext^1(L, \mathcal{O}_C) \cong Ext^1(L \otimes \mathcal{O}_C(D), \mathcal{O}_C \otimes \mathcal{O}_C(D))$ and compute a presentation of the module $N = H^0_*(C, E \otimes \mathcal{O}_C(D))$ as explained in Lemma 1.2;
(4) compute a presentation of $M' = N \otimes I_{D-B} = H^0_*(C, E \otimes \mathcal{O}_C(D)) \otimes I_{D-B}$;
(5) compute a presentation of $M = H^*_0(C, \widetilde{M}')$, which is the module $H^0_*(C, E \otimes \mathcal{O}_C(B))$ and proceed as explained in Lemma 1.3 to compute a set of generators of I_X. $\square$

Remark 1.6. Suppose that $|B|$ and $|L \otimes B|$ contain effective divisors $D \in |B|$ and $D' \in |L \otimes B|$, and that $\deg B > 2q - 2$ or $h^1(C, \mathcal{O}_C(B + jH)) = 0\ \forall j \geq 0$. Then in the proof of the theorem we can choose $D = B$, i.e. the module $M = H^0_*(C, E \otimes \mathcal{O}_C(B))$ can be directly obtained as an extension in $Ext^1_S((I_{D'})^*, (I_D)^*)$.

Remark 1.7. The theorem can be used to obtain examples of scroll surfaces by considering random effective divisors D, D' with fixed degrees such that $\deg D > 2q - 2$ and a random extension class in $Ext^1_S((I_{D'})^*, (I_D)^*)$. Defining L as the sheaf $\mathcal{O}_C(D' - D)$, the previous extension class determines an element in $Ext^1_S(L, \mathcal{O}_C) \cong H^1(C, L^*)$ and the condition $\deg D > 2q - 2 = 0$ ensures that every extension in $Ext^1_S(L, \mathcal{O}_C)$ can be obtained starting from an extension in $Ext^1_S((I_{D'})^*, (I_D)^*)$.

2. Some examples of "interesting" ruled surfaces

In this section we will construct some examples of "interesting" ruled surfaces by applying Theorem A, where by "interesting" we mean that these surfaces have some particular properties. We will use the computer-algebra system `Macaulay2` to execute all the computations described in the algorithm of Theorem A. In order to get the whole module $M = H^0_*(C, E \otimes \mathcal{O}_C(B))$, rather than the submodule M', we will use the corresponding implemented command in the computer-algebra system `Macaulay2`. If this command is not available in the computer-algebra system under use, the practical computation may have some difficulties, but, in any case, there is no problem when the assumptions in Remark 1.6 are valid because the assumptions allow us to choose $D = B$.

2.1. First example

Let C be a smooth curve of genus 2. Let E be a normalized rank 2 vector bundle of degree 2, so that we have the following exact sequence:

$$0 \to \mathcal{O}_C \to E \to L \to 0$$

where $L = \det(E) = c_1(E)$, $\deg L = 2$. Let B be any degree 3 divisor of C. On the surface $X = \mathbb{P}(E)$ we can consider the divisor $A = C_0 + p^*B \equiv C_0 + 3f$. A is a very ample divisor, whatever B is chosen, $h^0(X, A) = h^0(C, E \otimes B) = 6$ and it embeds X in $\mathbb{P}^5$ as a smooth scroll of degree 8 (see [16]); $g(X) = 2$. It is easy to see that X is 2-normal if and only if it is not contained in a quadric.

About this surface we have the following proposition (see [1]):

Proposition 2.1. *Let X be the surface above. X is contained in a rank 4 quadric cone whose vertex is a 4-secant line for X and therefore X is not 2-normal.*

Note that in [1] the proposition is proved by using geometric arguments and it is not considered the k-normality of X for $k \geq 3$ or the problem to determine a free resolution of the ideal I_X. Some more information about X can be found in [7], the article which suggested to us to approach the problem.

We now explicitly describe how to construct such an example by applying the algorithm in Theorem A. The first step is to get a projective model of a smooth curve C of genus 2. Following the method described in [21] for obtaining projective space models of general curves of genus 11, we choose a random space curve C of genus 2 and degree 5 by the function (see the proof of Lemma 6.3):

```
randomGenus2Curve = (R) -> (
     correctCodimAndDegree:=false;
     while not correctCodimAndDegree do (
          I=ideal syz transpose random(R^{-2,2:-3},R^{2:-4});
          correctCodimAndDegree=(codim I==2 and degree I==5););
     I);
```

We check the smoothness of C by means of the Jacobian criterion:

```
isSmoothSpaceCurve = (I) -> (
     singI:=I+minors(2,jacobian I);
     codim singI==4);
```

Proceeding with the construction of the starting data of the algorithm, we also need to choose t random points on C. This step can be performed by separating the points of a good hyperplane section (on a non-algebraically closed field it can happen that these points are not separated):

```
randomPoint = (C) -> (
     R:=ring C;
     isSinglePoint:=false;
     while not isSinglePoint do (
          hypsection:=C+ideal random(R^1,R^{-1});
          pt:=(decompose hypsection)#0;
          isSinglePoint=(degree pt==1););
     pt);
randomPoints = (C,t) -> (
     pt:=randomPoint C;i:=t-1;
     while i!=0 do (pti=randomPoint C;pt=intersect(pt,pti);i=i-1;);
     pt);
```

We are now ready to explicitly compute such an example by following the algorithm of Theorem A: we choose a random smooth genus 2 curve $C \in \mathbb{P}^3$ as explained above, we apply random choices for $D, D', N \in Ext^1(H^0_*(D'), H^0_*(D))$, B and, finally, we compute a set of generators of the ideal of X embedded via the linear system $|A| = |C_0 + p^*B|$ corresponding to our random choices.

The following input lines compute the ideal C of such a random curve C:

```
K=ZZ/101;
R=K[x_0..x_3]
C=randomGenus2Curve R
isSmoothSpaceCurve(C)
betti res C
```

We choose L and B as effective divisors of degrees 2 and 3 respectively. By Remark 1.6, we can therefore choose $D = B$. The points of an effective divisor are chosen via the function `randomPoints()`, which returns their ideals `Ldual` and `Ddual` in $\mathbb{P}^3$. Their intersection is called `D'dual`.

```
Ldual=randomPoints(C,2)
Ddual=randomPoints(C,3)
D'dual=intersect(Ldual,Ddual)
```

We then compute the S-modules $H^0_*(D)$ and $H^0_*(D')$, called resp. `DS` and `D'S`.

```
S=R/C
DSdual=substitute(Ddual,S);DS=Hom(DSdual,S);
D'Sdual=substitute(D'dual,S);D'S=Hom(D'Sdual,S);
```

We compute a presentation `phi` of a random module N in $Ext^1(H^0_*(D'), H^0_*(D))$ as explained in Lemma 1.2. For this purpose, we define the function `randomExt()`

```
randomExt = (A,B) -> (
     phia:=presentation A;
     phib:=presentation B;
     Homom:=Hom(image phia,coker phib);
     phiab:=homomorphism random(Homom,S^1);phiab=matrix phiab;
     phiNull:=0*random(target phia,source phib);
     phi:=(phia||phiab)|(phiNull||phib);
     coker phi)
```

and we apply this function to D'S and DS.

```
N=randomExt(D'S,DS)
```

As explained in the proof of Theorem A, the choice of N as an extension class in $Ext^1(H^0_*(D'), H^0_*(D))$ corresponds to the choice of a bundle E as an extension class in $Ext^1(L, \mathcal{O}_C)$ such that $N = H^0_*(C, E \otimes \mathcal{O}_C(D))$.

We should now compute $M' = N \otimes I_{D-B}$ and $M = H^0_*(\widetilde{M'})$. Here these two steps are not needed because of our choice $D = B$, and so $M = N = H^0_*(C, E \otimes \mathcal{O}_C(D))$. However, given a module N, the command to compute $H^0_*(\widetilde{N})$ is:

```
M=HH^0((sheaf N)(>=0));
```

We are now ready to compute an explicit set of generators of the ideal I_X of $X \subset \mathbb{P}^5$. We proceed as explained in the proof of Lemma 1.3 and we define for this purpose the function scrollIdeal():

```
scrollIdeal = (M) -> (
     phi=presentation prune image basis(0,M);
     T=K[y_1..y_(numgens target phi)];
     R:=ring phi;TR:=T**R;
     Phi:=substitute(phi,TR);
     IS=ideal(substitute(vars T,TR)*Phi);
     J:=saturate(IS, ideal substitute(vars R,TR));
     ideal mingens substitute(J,T))
```

Note that the first line computes a presentation of the submodule generated by the elements of degree 0 of M, i.e. by $H^0(C, E \otimes \mathcal{O}_C(B))$ (in this example this step is not needed by Remark 1.5 since $\deg C = 5$, hence M is generated in degree 0). The ideal I_X is called J in the script.

According to our random choices of $C, D = B, D'$, and N, the resulting surface $X \subset \mathbb{P}^5$ is computable by the following lines and has the following properties (a line beginning with oNN is the output line number NN of the program).

```
J=scrollIdeal(M)
dim J, degree J
o26 = (3, 8)
codim (J+minors(3,jacobian J))
o27 = 6
betti res J
o28 = total: 1 8 15 13 6 1
          0: 1 .  .  . . .
          1: . 1  .  . . .
          2: . 6  7  . . .
          3: . 1  8 13 6 1
```

In particular, X is a smooth surface of degree 6 and $\mathcal{I}_X$ has free resolution

$$0 \leftarrow \mathcal{I}_X \leftarrow \begin{array}{c} \mathcal{O}(-2) \\ \oplus \\ 6\mathcal{O}(-3) \\ \oplus \\ \mathcal{O}(-4) \end{array} \leftarrow \begin{array}{c} 7\mathcal{O}(-4) \\ \oplus \\ 8\mathcal{O}(-5) \end{array} \leftarrow 13\mathcal{O}(-6) \leftarrow 6\mathcal{O}(-7) \leftarrow \mathcal{O}(-8) \leftarrow 0.$$

Moreover, X is contained in only one quadric hypersurface Q, which is a rank 4 quadric cone having a 4-secant line L as vertex, according to Proposition 2.1:

```
Q=(gens J)_{0}
rank jacobian transpose jacobian Q
o30 = 4
singQ=ideal Q+ideal jacobian Q
L=saturate(singQ)
codim(L+J),degree(L+J)
o33 = (5, 4)
```

The k-normality of X can be investigated by computing the difference between the dimension of the degree k part of the coordinate ring of $X \subset \mathbb{P}^5$ and $h^0(X, \mathcal{O}_X(k)) = -1 + 4k^2 + 3k$. According to the notation in the script, the coordinate ring of X is T/J, and the function hilbertFunction(i,J) returns the dimension of its degree i part. Thus the following line computes the Hilbert function of the coordinate ring of X up to degree 10:

```
apply(0..10,i->hilbertFunction(i,J))
o34 = (1, 6, 20, 44, 75, 114, 161, 216, 279, 350, 429)
```

For $k = 1$ this difference is zero, while for $k = 2$ this difference is 1. Hence X is 1-normal but not 2-normal. Proceeding in this way, one can check that X is k-normal for any $k = 3, \dots, 10$. Since it is known that any surface of the type considered in this example is not 2-normal, but is k-normal for $k \geq 11$ (see [1]), the above example shows that the general surface of this type is in fact k-normal for $k \geq 3$.

Remark 2.2. Given the ideal of a non-degenerate surface $X \subset \mathbb{P}^r$ of degree d, it follows from the Castelnuovo bound that X is k-normal for $k \geq k_0 = d - 2 + r$, cf. [17]. The k-normality for $k < k_0$ can then be checked by computing the Hilbert function of X up to degree $k_0 - 1$.

2.2. Second example

Let C be a smooth curve of genus 1. Let E be a normalized rank 2 vector bundle of degree 0. Then we have one of the following cases:

(1) $E = \mathcal{O}_C \oplus \mathcal{O}_C$ and $\mathbb{P}(E) = C \times \mathbb{P}^1$
(2) $E = \mathcal{O}_C \oplus L$, where $L \neq \mathcal{O}_C$ but $\deg L = 0$
(3) E is given by the unique not trivial extension $0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$.

Let us call X_i for $i = 0, 1, 2$ the three surfaces $\mathbb{P}(E)$. It is known that, if we consider any degree 3 divisor B over C, X_i is embedded in $\mathbb{P}^5$ by $A = C_0 + p^*B \equiv C_0 + 3f$ as a smooth scroll surface of degree 6 (see [15]) and $g(X_i) = 1$. Moreover, $C_0 \simeq C$ is embedded as a smooth plane curve of degree 3 and $h^0(X_0, C_0) = 2$, $h^0(X_i, C_0) = 1$ for $i = 1, 2$.

About this surface we have the following proposition (see [2]):

Proposition 2.3. *Every X_i is projectively normal and it is contained exactly in only one net of quadrics $\Lambda_i \simeq \mathbb{P}^2$. Moreover:* (i) *$\Lambda_0$ contains only rank 4 quadrics whose line vertex is generically disjoint from X_0; in Λ_0 there is a smooth plane curve $\simeq C$ whose points correspond to the quadrics of Λ_0 whose vertex is contained in X_0.* (ii) *The generic quadric of Λ_1 is smooth; the only singular quadrics in Λ_1 have rank 4 and they are parametrized by a smooth plane curve $\mathcal{C} \simeq C$; the discriminant divisor in $\Lambda_1 \simeq \mathbb{P}^2$ is a reducible plane sextic $\mathcal{D} = 2\mathcal{C}$.* (iii) *The generic quadric of Λ_2 has rank 5; the only rank 4 quadrics in Λ_2 are parametrized by C_0: in fact their vertices are lines, tangent to C_0 with multiplicity 2.*

Since here $q = 1$, we can take a smooth plane cubic as C. Moreover, in order to satisfy the assumptions of Theorem A and Remark 1.6, we can choose $D = B$ and D' as effective divisors of degree 3, with $L = \mathcal{O}_C(D' - D)$. We then compute the modules $H^0_*(D)$ and $H^0_*(L \otimes \mathcal{O}_C(D))$, called resp. DS and D'S:

```
K=ZZ/101;
R=K[x_0..x_2]
C=ideal random(R^1,R^{-3})
codim (C+ideal jacobian C)
Ddual=randomPoints(C,3)
D'dual=randomPoints(C,3)
S=R/C
DSdual=substitute(Ddual,S);DS=Hom(DSdual,S);
D'Sdual=substitute(D'dual,S);D'S=Hom(D'Sdual,S)
```

2.2.1. Case 0

$E = \mathcal{O}_C \oplus \mathcal{O}_C$. Here $M = \mathtt{DS} \oplus \mathtt{DS}$:

```
M=DS++DS
```

and we proceed as in the previous example. $\mathcal{I}_{X_0}$ has free resolution:

$$0 \leftarrow \mathcal{I}_{X_0} \leftarrow 3\mathcal{O}(-2) \oplus 4\mathcal{O}(-3) \leftarrow 2\mathcal{O}(-3) \oplus 9\mathcal{O}(-4) \leftarrow 6\mathcal{O}(-5) \leftarrow \mathcal{O}(-6) \leftarrow 0.$$

The resolution of $\mathcal{I}_{X_0}$ shows that X_0 is contained in a net of quadrics, which we denote Λ_0, and it suggests that these quadrics have 2 independent linear relations among them. Indeed, it is straightforward to check that Λ_0 is the space generated by the 2×2 minors of the 3×2 matrix of linear forms given by the morphism $3\mathcal{O}(-2) \leftarrow 2\mathcal{O}(-3)$ appearing in the above resolution. Moreover, the ideal generated by Λ_0 defines a smooth scroll of dimension 3 and degree 4 in $\mathbb{P}^5$.

We verify that Λ_0 contains only rank 4 quadrics by checking that, in a space of parameters $\mathbb{P}^2$ for Λ_0, the locus where the 6×6 matrix representing the generic quadric has rank ≤ 4 (resp. ≤ 3) is the whole $\mathbb{P}^2$ (resp. empty).

Then it is possible to compute the vertex locus in $\mathbb{P}^2 \times \mathbb{P}^5$ of the net of quadrics Λ_0 and the locus $G \subset \mathbb{P}^2$ of the quadrics Γ_0 whose vertex line is contained in X_0, checking that indeed this is a smooth plane cubic. The fact that the curve G is isomorphic to C is a geometric consequence of the construction, since each vertex line is a line of the scroll X_0, which projects in a point of C.

2.2.2. Case 1

Here M is again a direct sum, namely $M = \mathtt{DS}\oplus \mathtt{D'S}$:

```
M=DS++D'S
```

and we proceed as in the previous example. $\mathcal{I}_{X_1}$ has free resolution:

$$0 \leftarrow \mathcal{I}_{X_1} \leftarrow 3\mathcal{O}(-2) \oplus 2\mathcal{O}(-3) \leftarrow 9\mathcal{O}(-4) \leftarrow 6\mathcal{O}(-5) \leftarrow \mathcal{O}(-6) \leftarrow 0.$$

As in the previous subcase, we can compute the matrix representing a generic quadric in the net Λ_1 and the discriminant divisor. Then we can check that it is indeed the square of a cubic form G and that the singular quadrics, parametrized by G, all have rank 4.

We want to point out that, for all the constructed examples of surfaces X_1, we obtained the following nice geometric configuration, not shown by Proposition 2.3 and completely unexpected. The locus Y_1 of the lines in $\mathbb{P}^5$, which are vertices of the singular quadrics in the net Λ_1 of quadrics containing X_1, is again a geometrically ruled surface of degree 6. According to the classification in Proposition 2.3, Y_1 is of the same type as X_1. Moreover, the intersection $X_1 \cap Y_1$ consists of two plane cubic curves lying in disjoint planes, one of them being C_0, the tautological divisor of $\mathbb{P}(E)$.

In order to compute C_0, we start by defining a function to compute the ideal of the fiber in $\mathbb{P}^5$ of an effective divisor over C.

```
pullbackIdeal = (I) -> (
     R:=ring I;TR:=ring IS;
     J:=substitute(I,TR)+IS;
     J=saturate(J,ideal substitute (vars R, TR));
     ideal mingens substitute(J,T))
```

With this function we compute the ideal `H` corresponding to the divisor $p^*(D)$ (3 lines). The ideal `C0` of C_0 may be computed as the quotient of the ideal of a hyperplane section of X_1 containing $p^*(D)$ by `H`, the ideal of $p^*(D)$:

```
H=pullbackIdeal(Ddual)
C0=(ideal H_0+J):H
```

2.2.3. Case 2

Here M is an extension in $Ext^1(\mathtt{DS},\mathtt{DS})$, where `DS` is the module constructed as in Case 0 corresponding to an effective divisor D of degree 3. We apply the function `randomExt()` defined in Section 2.1:

```
M=randomExt(DS,DS)
```

Then we proceed as in the previous example. $\mathcal{I}_{X_2}$ has free resolution:

$$0 \leftarrow \mathcal{I}_{X_2} \leftarrow 3\mathcal{O}(-2) \leftarrow 2\mathcal{O}(-3) \leftarrow 9\mathcal{O}(-4) \leftarrow 6\mathcal{O}(-5) \leftarrow \mathcal{O}(-6) \leftarrow 0.$$

As in the previous case, we can compute the matrix representing a generic quadric in the net Λ_2, we can check that all the quadrics have rank ≤ 5, and we can compute the divisor of the rank 4 quadrics in Λ_2, a smooth cubic G.

According to Proposition 2.3, it is possible to check that the vertex of any rank 4 quadric in Λ_2 is a line tangent to X_2 at a point of C_0.

Again, there is a nice geometric configuration, not shown by Proposition 2.3 and completely unexpected. The locus Y_2 of the lines in $\mathbb{P}^5$, which are vertices of the singular quadrics in the net Λ_2 of quadrics containing X_2, is a geometrically ruled surface of degree 6. According to the classification in Proposition 2.3, Y_2 is of the same type as X_2. Moreover, the intersection $X_2 \cap Y_2$ consists of the cubic C_0 counted twice. Therefore a vertex of a quadric in Λ_2 is a line L tangent to X_2 at the point of C_0 given by the intersection of L with C_0.

3. Embeddings with fibers of higher degree

In this section we develop a method to construct explicitly the ideals of surfaces which are $\mathbb{P}^1$-bundles over a smooth curve C, embedded in such a way that every fiber is a rational curve of degree $k \geq 2$. In other words, we consider polarized surfaces (X, A), where $X := \mathbb{P}(E)$ as in Section 1 and $A = kC_0 + p^*B$ is very ample. We aim to prove the following:

Theorem B (char $\mathbb{K} = 0$ *or* $q \leq 1$). *Let $C \subset \mathbb{P}^m$ be a smooth curve C of genus q, B a divisor on C and L a line bundle over C. Consider a normalized rank 2 vector bundle $E \in Ext^1(L, \mathcal{O}_C)$ over C given by an extension $0 \to \mathcal{O}_C \to E \to L \to 0$ and suppose that, for some positive integer k, the divisor $A = kC_0 + p^*B$ on the surface $X = \mathbb{P}(E)$ is very ample.*

Then there is an algorithm yielding a set of generators of the ideal I_X of the embedded X in $\mathbb{P}^{h^0(X,A)-1} = \mathbb{P}(H^0(X, A)^*)$ by $|A|$.

A description of the explicit form of the data required by the algorithm is given in the introduction, immediately after the statement of the main theorem.

Let us say a few words about the algorithm which we are going to present. We adopt here the choice to proceed in a geometric way as clear and straightforward as is possible rather than to optimize the computational aspect. This choice allows us an easy implementation, but it may be computationally not a good one: for example, the computational complexity could be a problem for large values of k (see Remark 3.5). We also point out that the restriction on char $\mathbb{K} = 0$ (if $q > 1$) can be bypassed with a computational check (see Remark 3.6).

Firstly, let us develop a technical criterion needed in the following. We recall the following theorem of Butler:

Theorem 3.1 (*Bu, Thm. 5.1A*). *(char $\mathbb{K} = 0$ or $q \le 1$). Let E be a vector bundle over a smooth projective curve C of genus q, $p : E \to C$ the projection, and let $X = \mathbb{P}(E)$. If Z is a (-1) p-regular line bundle over X with $\mu^-(p_*Z) > 2q$, then Z is normally generated.*

We will use this result in the form of the following corollary:

Corollary 3.2. *(char $\mathbb{K} = 0$ or $q \le 1$). Let E be a vector bundle over a smooth projective curve C of genus q, $p : E \to C$ the projection, and let $X = \mathbb{P}(E)$. If $\mu^-(E) > 2q$, the tautological divisor τ of X is very ample and $X \subset \mathbb{P}(H^0(X, \tau)^*)$ is projectively normal.*

Proof. It is well known that the condition $\mu^-(E) > 2q$ implies that the tautological divisor τ of X is very ample, cf. Lemma 1.12 of [6]. Since τ is very ample, it is enough to prove that X is normally generated in $\mathbb{P}(H^0(X, \tau)^*)$.

For this we apply Theorem 3.1. We recall that a divisor Z is called (-1) p-regular if, for every fiber F of p over C, $H^i(F, Z_{|F}(-1-i)) = 0$, for all $i > 0$. In our case these groups are $H^i(\mathbb{P}^h, \mathcal{O}_{\mathbb{P}^h}(-i)) = 0$, where $h = \operatorname{rk} E - 1$, hence τ is automatically (-1) p-regular. For the second condition of Theorem 3.1, we remark that $\mu^-(p_*\tau) = \mu^-(E)$. □

Now we are ready to give the required criterion.

Proposition 3.3. *(char $\mathbb{K} = 0$ or $q \le 1$). Let E be a vector bundle over a (smooth) curve C of genus q and let D be an effective divisor of degree d on C. If the condition*

$$\mu^-(E) + d > 2q \tag{3.1}$$

is satisfied, the divisor $C_0 + p^*(D)$ is very ample on $X = \mathbb{P}(E)$ and the image X' of X, given by the linear system $|C_0 + p^*D|$, is projectively normal.

Proof. Just apply Corollary 3.2 to $E' := E \otimes \mathcal{O}_C(D)$ and recall that $\mu^-(E') = \mu^-(E) + d$. □

Remark 3.4. Condition (3.1) is not an evident numerical condition, since it is not clear how to compute $\mu^-(E)$ for a given vector bundle E.

However, if the genus q of the curve C satisfies $q \ge 2$, the set of points in $Ext^1(L, \mathcal{O}_C)$ parametrizing a semi-stable vector bundle E is a Zariski open set, see the classical theorem [19, Thm. 2]. Hence for a general choice of such an extension the corresponding E is semi-stable and $\mu^-(E) = \mu(E) = -e/2$.

For the case $q = 1$, it is known that if E is indecomposable then E is semi-stable. If instead E is decomposable, say $E = L \oplus L'$, then $\mu^-(E) = \min(\deg(L), \deg(L'))$, see [3, Lemma 2.8]. In the case $q = 0$, E is necessarily of the type $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ and $\mu^-(E) = \min(a, b)$.

Proof of Theorem B. Choose an effective divisor D on C of degree d such that $D-B$ is effective, $|L \otimes \mathcal{O}_C(D)| \ne \emptyset$, and satisfying conditions (1.3) and (3.1). Note that this is always possible by choosing the degree of D big enough. By Proposition 3.3, the divisor $C_0 + p^*(D)$ on the surface $X = \mathbb{P}(E)$ is very ample. Let $X' \subset \mathbb{P}^r$ be the image of the embedding $\iota : X = \mathbb{P}(E) \hookrightarrow \mathbb{P}^r$ given by $|C_0 + p^*(D)|$, where $r = h^0(C_0 + p^*(D)) - 1$. By applying Theorem A (with the same choice of D), we can obtain a set of generators of the ideal $I_{X'}$ of $X' \subset \mathbb{P}^r$.

Let R be a section of the sheaf $i_*p^*\mathcal{O}_C(kD - B)$ and let H be the hyperplane divisor of $\mathbb{P}^r$. Since R is an effective divisor of X' we have

$$0 \to \mathcal{I}_{X'} \to \mathcal{I}_R \to \mathcal{I}_{R,X'} \to 0,$$

where $\mathcal{I}_{R,X'}$ is the relative ideal sheaf of R in X'.

Notice that the surface X' is projectively normal by Proposition 3.3. Therefore $H^1(\mathbb{P}^r, \mathcal{I}_{X'}(kH)) = 0$ and the above sequence, tensorized with $\mathcal{O}_{\mathbb{P}^r}(kH)$, gives

$$0 \to H^0(\mathbb{P}^r, \mathcal{I}_{X'}(kH)) \to H^0(\mathbb{P}^r, \mathcal{I}_R(kH)) \to H^0(\mathbb{P}^r, \mathcal{I}_{R,X'}(kH)) \to 0.$$

Since $\mathcal{I}_{R,X'}(kH) = \mathcal{O}_{X'}(kH - R) \cong \mathcal{O}_X(kC_0 + p^*(B)) = \mathcal{O}_X(A)$, we have

$$H^0(X, A) \cong \frac{H^0(\mathbb{P}^r, \mathcal{I}_R(kH))}{H^0(\mathbb{P}^r, \mathcal{I}_{X'}(kH))}.$$

Let $f_0, \ldots, f_n$ be a set of polynomials in $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k))$ whose equivalence classes form a basis of this quotient space ($n = h^0(X, A) - 1$). The image of X under the linear system $|A|$ is then given in the following way: if $y_0, \ldots y_n$ are indeterminates and V is the coordinate ring of $X' \subset \mathbb{P}^r$, the ideal I_X is the kernel of the map $\mathbb{K}[y_0, \ldots, y_n] \to V$ obtained by sending y_i to $[f_i]$, the class of f_i in V.

The algorithm based on the above considerations is therefore the following:

(1) choose on $C \subset \mathbb{P}^m$ an effective divisor D satisfying conditions (1.3) and (3.1) and such that $D - B$ is effective, $|L \otimes \mathcal{O}_C(D)| \neq \emptyset$;
(2) choose a divisor $D' \in |L \otimes \mathcal{O}_C(D)|$ and, from the ideal of C and from L, compute S and a set of generators of the ideals I_D and $I_{D'}$ (as S-modules);
(3) identify the extension in $Ext^1_S(I_{D'}, I_D)$ corresponding to the given extension of E as an element in $Ext^1(L, \mathcal{O}_C) \cong Ext^1(L \otimes \mathcal{O}_C(D), \mathcal{O}_C \otimes \mathcal{O}_C(D))$ and compute a presentation of the module $N = H^0_*(C, E \otimes \mathcal{O}_C(D))$ as explained in Lemma 1.2;
(4) proceed as explained in Lemma 1.3 to compute a set of generators of I'_X, where X' is the embedding of $\mathbb{P}(E)$ by $|C_0 + p^*(D)|$;
(5) for each point P' (with multiplicity) of a fixed element of $|kD - B|$, compute the ideal of the fiber of X' over such point. This ideal can be computed in the same way used to compute I'_X, as explained in Lemma 1.3, but adding to the relations, besides the ones coming from the presentation of N, also the generators of the ideal of the point P' (with multiplicity);
(6) the intersection of all the ideals computed in the previous step gives the ideal I_R of the divisor $R \in |i_* p^* \mathcal{O}_C(kD - B)|$;
(7) compute a set of homogeneous polynomials of degree k $f_0, \ldots, f_n$ whose equivalence classes form a basis of $H^0(\mathbb{P}^r, \mathcal{I}_R(kH))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(kH))$;
(8) compute the ideal of X as the kernel of the map $\mathbb{K}[y_0, \ldots, y_n] \to S$ which sends y_i to the equivalence class of f_i in the coordinate ring S of X'. $\square$

Remark 3.5. In the previous algorithm, a difficult point is to compute the system of hypersurfaces of degree k in $\mathbb{P}^r$ through $\deg(kD - B)$ lines of the scroll X'. We do not know how hard is this task computationally when k or the degree of D increases.

Remark 3.6. The assumption char $\mathbb{K} = 0$ (if $q > 1$) of Theorem B can be replaced by a computational check.

Proof. This assumption ensures that the scroll X', corresponding to a choice of D as in the proof of Theorem B, is k-normal. Therefore the assumption is not required if X' is k-normal.

Once the auxiliary divisor D is chosen and the resulting surface X' is determined, the k-normality of X' can be computationally checked. If X' is not k-normal, change D and repeat the check. $\square$

Remark 3.7. Even without the very ampleness assumption on $A = kC_0 + p^*B$, the algorithm of Theorem B still holds for computing the ideal of the image of X by the rational map $\phi_{|A|}$ associated to $|A|$ (even for $k = 1$).

Proof. Perform steps (1)–(7) of the algorithm described in Theorem B and compute a set of homogeneous polynomials $f_0, \ldots, f_n$ of degree k whose equivalence classes form a basis of $H^0(X, A) \cong H^0(\mathbb{P}^r, \mathcal{I}_R(kH))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(kH))$: on X' the map associated to $|A|$ corresponds to the rational map $(f_0 : \ldots : f_n)$ and step (8) computes the ideal of the image of X by $\phi_{|A|}$. $\square$

Remark 3.8. Given a divisor $A = kC_0 + p^*B$, it is possible to decide computationally whether this divisor is very ample.

Proof. As explained in the above remark, apply the algorithm of Theorem B and compute the image of X, say Y, by the rational map $\phi_{|A|}$ associated to $|A|$. The point is to determine computationally whether $\phi_{|A|}$ is an embedding.

This can be done in the following way: $\phi_{|A|}$ is an embedding if and only if Y is a smooth surface of degree A^2 and $|A|$ has empty base locus. Indeed, assume that Y is smooth of degree A^2 and that $\phi_{|A|}$ is a morphism. Then $\phi_{|A|}$ is a birational morphism and, since X is a minimal surface being geometrically ruled, a posteriori $\phi_{|A|}$ is an isomorphism (the case $X = \mathbb{F}_{1,0}$ has to be considered separately).

These conditions can easily be checked computationally: the base locus of A is determined by the vanishing set of $(f_0, \ldots, f_n)$ on X', while the dimension, smoothness, and degree of Y are directly computable from its ideal. $\square$

4. Examples of ruled surfaces with higher degree fibers

4.1. An example of conic bundles

Let C be a smooth curve of genus 1. Let E be a normalized rank 2 vector bundle of degree 1 which is given by the only non-trivial extension:

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(P) \to 0$$

where $\det(E) = c_1(E) = \mathcal{O}_C(P)$ and P is a fixed point of C. Let Q be any other point of C, eventually $Q = P$. It is known that, on the surface $X = \mathbb{P}(E)$, the divisor $A = 2C_0 + p^*Q \equiv 2C_0 + f$ is very ample, whatever Q is chosen, $h^0(X, A) = h^0(C, S^2(E) \otimes \mathcal{O}_C(Q)) = 6$ and A embeds X in $\mathbb{P}^5$ as a smooth ruled surface of degree 8, whose fibers are embedded as smooth plane conics (see [15]); $g(X) = 3$. In this case, C_0 is embedded as a smooth plane cubic and $h^0(X, C_0) = 1$.

We now apply the algorithm of Theorem B to compute a set of generators of the ideal I_X. We set $B = Q$ and we can choose $D = Q + Q'$, where Q' is a further point, so that $D - B$ is effective and $2D - B = Q + 2Q'$.

We fix a choice of a smooth plane cubic curve $C \subset \mathbb{P}^2$, named C, and of three points p, q, q', which correspond respectively to the three points P, Q, Q' on C.

Let S be coordinate ring of C. The necessary steps to compute a presentation of the S-module $M = H^0_*(C, E \otimes \mathcal{O}_C(D))$ are as usual:

```
K=QQ;
R=K[x_0..x_2]
C=ideal (x_0*(x_2)^2-x_1*(x_1+x_0)*(x_1+2*x_0))
p=ideal (x_1,x_2);q=ideal (x_1,x_0);q'=ideal (x_1+x_0,x_2);
Ddual=intersect(q,q');
D'dual=intersect(p,Ddual)
S=R/C
DSdual=substitute(Ddual,S);DS=Hom(DSdual,S);
D'Sdual=substitute(D'dual,S);D'S=Hom(D'Sdual,S);
M=randomExt(D'S,DS)
```

We then compute the ideal J of $X' \subset \mathbb{P}^4$, the embedding of $\mathbb{P}(E)$ by the linear system $|C_0 + p^*(D)|$ as explained in Section 2:

```
J=scrollIdeal(M) --ideal of X'
```

X' is a smooth surface (of degree 5), and $\mathcal{I}_{X'}$ has free resolution:

$$0 \leftarrow \mathcal{I}_{X'} \leftarrow 5\mathcal{O}(-3) \leftarrow 5\mathcal{O}(-4) \leftarrow \mathcal{O}(-5) \leftarrow 0.$$

The next step is to compute I_R, where R is the pullback on X' of the divisor $2D - B = Q + 2Q' \in \mathrm{Div}(C)$.

```
q'squareFiber=pullbackIdeal(q'^2)
qFiber=pullbackIdeal(q)
A=intersect(q'squareFiber,qFiber); betti A
o30 = generators: total: 1 7
                      0: 1 .
                      1: . 6
                      2: . 1
```

We now perform step (7) of the algorithm in the proof of Theorem B, i.e., we compute a set of homogeneous polynomials of degree 2 $f_0, \dots, f_n$ whose equivalence classes form a basis of the quotient space $H^0(\mathbb{P}^r, \mathcal{I}_R(2H))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(2H))$. The resolution of I'_X shows that X' is not contained in any quadric. Therefore, in this case, the vector space $H^0(\mathbb{P}^r, \mathcal{I}_R(2H))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(2H))$ is just $H^0(\mathbb{P}^4, \mathcal{I}_R(2H))$, the space of quadrics of $\mathbb{P}^4$ passing through the fibers over $Q+2Q'$:

```
Q=super basis(2,A) --the linear system |2H-2D+B|
```

Finally, we compute X as the image of X' via the embedding given by the linear system of quadrics Q obtained above:

```
Z=K[z_0..z_5];S'=T/J;f=map(S',Z,substitute(Q,S'))
I=ker f --ideal of X
```

The ideal I will be the ideal of the surface X. It is straightforward to check that $X \subset \mathbb{P}^5$ has degree 8 and $\mathcal{I}_X$ has resolution:

$$0 \leftarrow \mathcal{I}_X \leftarrow \mathcal{O}(-2) \oplus 8\mathcal{O}(-3) \leftarrow 15\mathcal{O}(-4) \leftarrow 8\mathcal{O}(-5) \leftarrow \mathcal{O}(-6) \leftarrow 0.$$

The Hilbert function of X is given by:

$$h^0(X, \mathcal{O}_X(t)) = 1, 6, 20, 42, 72, 110, 156, 210, 272, 342 \dots \quad \text{for } t = 0, 1, 2 \dots.$$

4.2. A ruled surfaces with cubic fibers

Let C be a smooth curve of genus 1. Let E be a normalized rank 2 vector bundle of degree 1 which is given by the only non-trivial extension:

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(P) \to 0$$

where $\det(E) = c_1(E) = \mathcal{O}_C(P)$ and P is a fixed point of C. On the surface $X = \mathbb{P}(E)$, the divisor $A = 3C_0$ is very ample: the numerical criterion of Reider is satisfied (cf. [20]). $h^0(X, A) = h^0(C, S^3(E)) = 6$ and A embeds X in $\mathbb{P}^5$ as a smooth ruled surface of degree 9, whose fibers are embedded as twisted cubics (see [15]); $g(X) = 4$. In this case, C_0 is embedded as a smooth plane cubic as $h^0(X, 2C_0) = 3$.

We have $B = 0$ and we can choose $D = Q + Q'$, where Q, Q' are any couple of points. The divisor $3D - B$ is then $3D = 3Q + 3Q'$.

We fix a choice of a smooth plane cubic curve $C \subset \mathbb{P}^2$, named C, and of three points p, q, q', which correspond respectively to the three points P, Q, Q' on C. Let S be coordinate ring of C. The necessary steps to compute a presentation of the S-module $M = H^0_*(C, E \otimes \mathcal{O}_C(D))$ are as usual. For shortness, we perform the same choices as in Section 4.1 and we start with the ideal J.

We now compute the ideal I_R, where R is the pullback on X' of the divisor $3D - B = 3Q + 3Q' \in \mathrm{Div}(C)$:

```
qcubeFiber=pullbackIdeal(q^3)
q'cubeFiber=pullbackIdeal(q'^3)
A=intersect(qcubeFiber,q'cubeFiber); betti A
o42 = generators: total: 1 11
                      0: 1  .
                      1: .  .
                      2: . 11
```

Since the ideal J of X' contains a five-dimensional space of cubics, we need to find a set of representative cubics Q for a basis of the quotient space $H^0(\mathbb{P}^r, \mathcal{I}_R(3H))/H^0(\mathbb{P}^r, \mathcal{I}_{X'}(3H))$:

```
a3=super basis(3,A)
j3=super basis(3,J)
Q=super basis (3,ideal a3/ideal j3)
Q=matrix(T,entries Q)
```

Finally, we compute X as the image of X' via the embedding given by the linear system of cubics Q obtained above:

```
Z=K[z_0..z_5];S'=T/J;f=map(S',Z,substitute(Q,S'))
I=ker f --ideal of X
```

The ideal I will be the ideal of the surface X. It is straightforward to check that $X \subset \mathbb{P}^5$ has degree 8 and $\mathcal{I}_X$ has resolution:

$$0 \leftarrow \mathcal{I}_X \leftarrow 11\mathcal{O}(-3) \leftarrow 18\mathcal{O}(-4) \leftarrow 9\mathcal{O}(-5) \leftarrow \mathcal{O}(-6) \leftarrow 0.$$

The Hilbert function of X is given by:

$$h^0(X, \mathcal{O}_X(t)) = 1, 6, 21, 45, 78, 120, 171, 231, 300, 378, 465 \ldots \quad \text{for } t = 0, 1, 2 \ldots.$$

Remark 4.1. The projective normality of the previous surface is proved in the rather long Proposition 4.6 of [4]. As a preparatory lemma, in [4] the authors proved that the projective normality of X is equivalent to the 2-normality of X, which is equivalent to the fact that X does not lie on any quadric (see Lemma 4.3 of [4]). This property can be immediately verified for any constructed example, by looking at the resolution of I_X.

5. Some varieties related to conic bundles

5.1

Let D be a degree d effective divisor of C satisfying (3.1) and $E' = E \otimes \mathcal{O}_C(D)$. We want to describe shortly some varieties related to $\mathbb{P}(E')$ and their geometric correlations.

Let us define $E_1 := S^2(E')$ and let us consider the image X_1 of the 3-fold $\mathbb{P}(E_1)$ in $\mathbb{P}^s := \mathbb{P}(H^0(C, E_1)^*) \subset \mathbb{P}^N$ via the linear system given by the tautological divisor T_1 of E_1: as D satisfies (3.1), it is easy to see that $S^2(E')$ is very ample by Lemma 1.12 of [6], because $\mu^-(S^2(E')) = 2\mu^-(E') > 4q$.

On the other side, let $X' \subset \mathbb{P}^r$ be the image of $\mathbb{P}(E')$ embedded via its tautological bundle, i.e. via $|C_0 + p^*(D)|$, which is very ample because D satisfies (3.1). Let X'' be the image of X' under the 2-Veronese embedding $\nu : \mathbb{P}^r \hookrightarrow \mathbb{P}^N$, where $N = \binom{r+2}{2} - 1$, i.e. the image of X under the composition $\nu \circ \iota$ where ι is the embedding of X in $\mathbb{P}^r$. Then X'' is the image of X via the map associated to the linear system $|2C_0 + p^*(2D)|$. Algebraically, the map is given as follows. Let $y_0, \ldots, y_r$ be a basis of $H^0(C, E')$: then $y_0^2, y_0y_1, \ldots, y_r^2$ is a basis of $S^2(H^0(C, E'))$ and, defining $z_{i,j}$ for $i \leq j$ as a set of coordinates for $\mathbb{P}^N$, the composition $\nu \circ \iota$ is given by mapping $z_{i,j}$ to the product y_iy_j, considered as an element in $\mathcal{O}_{\mathbb{P}(E')}(2)$.

We want to describe a method to compute a set of generators of the ideal of the image X_1 of $\mathbb{P}(E_1)$ and also of the ideal of the image X'' of $\mathbb{P}(E)$. Notice that X'' is a surface contained in X_1. Indeed, the projective normality of X' (see the proof of Theorem B) implies the exactness of the sequence

$$0 \to H^0(\mathbb{P}^r, \mathcal{I}_{X'}(2)) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \to H^0(X', \mathcal{O}_{X'}(2)) \to 0, \tag{5.1}$$

where $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \cong S^2(H^0(C, E'))$ and $H^0(X', \mathcal{O}_{X'}(2)) \cong H^0(C, S^2(E'))$; hence $S^2(H^0(C, E'))$ surjects to $H^0(C, S^2(E'))$.

For sake of simplicity, let us now assume that $C \subset \mathbb{P}^m$ satisfies

$$\deg(C) \geq 2q + 1, \tag{5.2}$$

so that C is projectively normal too and its coordinate ring is $S = \oplus_{t \geq 0} H^0(C, \mathcal{O}_C(t))$.

Then $H^0_*(C, E')$ is generated by $H^0(C, E')$, as a graded module over S. In fact, by the above assumptions, we have $\deg(\mathcal{O}_C(t)) \geq 2q$ for $t \geq 1$ and $\mu^-(E') > 2q$, and the surjectivity of the natural map $H^0(C, E') \otimes H^0(C, \mathcal{O}_C(t)) \to H^0(C, E'(t))$ is an application of Theorem 2.1 of [6]. In the same way $H^0(C, S^2(E'))$ generates the module $H^0_*(C, S^2(E'))$.

We now describe how to determine the equations of X_1. As the first step, notice that $S^2(H^0_*(C, E'))$ surjects to $H^0_*(C, S^2(E'))$. Indeed, since $S^2(H^0(C, E'))$ surjects to $H^0(C, S^2(E'))$ and $H^0(C, S^2(E'))$ generates $H^0_*(C, S^2(E'))$ by (5.2), the images of $y_0^2, y_0y_1, \dots, y_r^2$ in $H^0_*(C, S^2(E'))$ form a set of generators of this module.

Let the following one be a free presentation of the S-module $H^0_*(C, E')$:

$$M_2 \to M_1 \to H^0_*(C, E') \to 0. \tag{5.3}$$

Then $S^2(H^0_*(C, E'))$ has a free presentation

$$(M_2 \otimes M_1) \xrightarrow{\phi} S^2M_1 \to S^2(H^0_*(C, E')) \to 0, \tag{5.4}$$

where, according to the notation introduced at the beginning of this subsection, S^2M_1 is the free S-module with generators $z_{i,j}$ for $i \leq j$ and the map $S^2M_1 \to S^2(H^0_*(C, E'))$ is determined by sending $z_{i,j}$ to the element corresponding to y_iy_j in $S^2(H^0_*(C, E'))$.

We now compute the polynomials in $\mathbb{K}[z_{i,j}]$ which are zero in $S(H^0_*(C, S^2E'))$. Let N be the set of elements $\sum \alpha_{i,j}z_{i,j} \in S^2M_1$ ($\alpha_{i,j} \in \mathbb{K}$) such that $\sum \alpha_{i,j}y_iy_j \in H^0(\mathbb{P}^r, I_{X'}(2))$. By (5.1), the symmetric algebra $S(H^0(C, S^2(E')))$ is isomorphic to the symmetric algebra $S(V)$, where V is the vector space $S^2(H^0(C, E'))/N$. Hence a polynomial $g(z_{i,j}) \in S(S^2M_1)$ whose image is non-zero in $S(S^2(H^0_*(C, E')))$ is zero in $S(H^0_*(C, S^2(E')))$ if and only if g is in the ideal generated by N.

A basis of N over $\mathbb{K}$ can be computed in the following way. By following our algorithm we compute $I_{X'}$ and a basis of $H^0(\mathbb{P}^r, I_{X'}(2))$, and for each element $\sum \alpha_{i,j}y_iy_j$ (written with $i \leq j$) in this basis we consider the corresponding element $\sum \alpha_{i,j}z_{i,j} \in S^2M_1$: the set of all these elements forms a basis of N.

We are therefore able to get a set of generators of the ideal of $X_1 \subset \mathbb{P}^N$ in the following way. In the ring $\mathbb{K}[z_{i,j}, x_0, \dots, x_m]$ we multiply the matrix associated to the map ϕ in (5.4) with a row vector given by the variables $z_{i,j}$ (by considering their exact order), and we consider the ideal generated by these elements and by the linear forms in N. Then we saturate this ideal with respect to the irrelevant ideal $(x_0, \dots, x_m)$ and we consider the intersection of this ideal with the subring $\mathbb{K}[z_{i,j}]$. Note that X_1 is degenerate if $H^0(\mathbb{P}^r, I_{X'}(2)) \neq 0$: X_1 lies in the $\mathbb{P}^s \subset \mathbb{P}^N$ given by the linear equations coming from N, obtained from $H^0(\mathbb{P}^r, I_{X'}(2))$.

The ideal of the surface X'' can be obtained by adding to the ideal of X_1 the forms of degree 2 (in the $z_{i,j}$) lying in the kernel of the map

$$S^2(S^2(H^0(C, E'))) \to S^4(H^0(C, E')).$$

Indeed, these forms generate the ideal of the Veronese image of $\mathbb{P}^r$ in $\mathbb{P}^N$.

In terms of sheaves on C, we have the following exact sequence:

$$0 \to \mathcal{L} \to S^2(S^2(E')) \to S^4(E') \to 0,$$

where $\mathcal{L}$ is a line bundle, E' being of rank 2. The quadratic forms obtained above are global sections of $\mathcal{L}$. We call these quadrics the *relative Veronese quadrics*, since they give fiberwise the ideal of the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^2$.

Remark 5.1. If $\deg(C) \leq 2q$ and $C \subset \mathbb{P}^m$ is not projectively normal, then we can still find explicit equations of X_1, by arguing as already done in the proof of Lemma 1.3. Indeed, instead of the presentation (5.3), we can take a presentation of the submodule $M' \subset H^0_*(C, E')$ generated by $H^0(C, E')$ and the corresponding presentation ϕ of the submodule generated by $S^2(H^0(C, E'))$ in $S^2(H^0_*(C, E'))$. Since X' is projectively normal, $S^2(H^0(C, E'))$ surjects to $H^0(C, S^2(E'))$ and we can proceed as above.

5.2. An alternative algorithm for conic bundles

From the short exact sequence $0 \to \mathcal{O}_C \to E \to L \to 0$ one can derive two other short exact sequences where E_1 fits, namely:

$$0 \to E \otimes \mathcal{O}_C(2D) \to E_1 \to L^2 \otimes \mathcal{O}_C(2D) \to 0 \tag{5.5}$$

and

$$0 \to \mathcal{O}_C(2D) \to E_1 \to E \otimes L \otimes \mathcal{O}_C(2D) \to 0. \tag{5.6}$$

In fact, we have $0 \to E \otimes \mathcal{O}_C \to S^2(E) \to S^2(L) \to 0$, which can be rewritten as $0 \to E \to S^2(E) \to L^2 \to 0$ and from this sequence we can proceed in two ways: either we tensorize with $\mathcal{O}_C(2D)$ and we get (5.5), or we dualize it and we get $0 \to L^{-2} \to S^2(E^*) \to E^* \to 0$, where L^{-i} denotes the ith tensor power of L^*. Since $E^* \cong E \otimes L^*$, we obtain (5.6) by tensorizing with $L^2 \otimes \mathcal{O}_C(2D)$.

Here we want to use (5.6). If $\mu^-(E) + 2d > 2q - e$, then by Lemma 1.12 of [6] $E \otimes L \otimes \mathcal{O}_C(2D)$ is very ample and its tautological bundle τ defines an embedding Σ of $\mathbb{P}(E)$. The sequence (5.6) implies both $T_{1\Sigma} = \tau$ and $\Sigma \in |T_1 - p_1{}^*(2D)|$, where p_1 is the projection of X_1 to C (X_1 is the image of $\mathbb{P}(E_1)$ by the tautological divisor T_1 of E_1, see the beginning of Section 5.1). If furthermore $d > q - 1$, then $h^1(C, 2D) = 0$ and Σ is linearly normal in X_1.

We can also proceed by using Σ to compute a set of generators of the ideal I_X, where, as usual, X denotes the image of $\mathbb{P}(E)$ via the map associated to the linear system $|A|$ in Theorem B for the case $k = 2$. In this case, further conditions on D are needed. Let us suppose that D satisfies, besides the (already required) conditions $\mu^-(E)+d > 2q$, $\mu^-(E)+2d > 2q-e$ and $d > q - 1$, the further condition

$$H^0(C, L^2 \otimes \mathcal{O}_C(4D - B)) \neq 0. \tag{5.7}$$

Consider a divisor $D' \in |L \otimes \mathcal{O}_C(2D)|$, so that $\tau = C_0 + p^*(D')$. Then the sheaf $\mathcal{O}_{X_1}(2T_1 - p_1^*(2D' - B))$ on X_1 restricts to Σ to the sheaf

$$\mathcal{O}_\Sigma(2\tau - p^*(2D' - B)) \cong \mathcal{O}_X(2(C_0 + p^*D') - p^*(2D' - B)) \cong \mathcal{O}_X(2C_0 + p^*B) = \mathcal{O}_X(A).$$

Since $\mu^-(E_1) = 2\mu^-(E) + 2d > 4q$, X_1 is projectively normal by Corollary 3.2. Hence, if

$$|2T_1 - p_1^*(2D' - B)| \to |2\tau - p^*(2D' - B)| \text{ is surjective,} \tag{5.8}$$

we can proceed as in the proof of Theorem B: we take an effective divisor $R_1 \in |p_1^*(2D' - B)|$ and we have

$$H^0(X, \mathcal{O}_X(A)) \cong \frac{H^0(\mathbb{P}^s, \mathcal{I}_{R_1}(2H_1))}{H^0(\Sigma, \mathcal{I}_\Sigma(2H_1))},$$

where H_1 denotes a hyperplane of $\mathbb{P}^s \subset \mathbb{P}^N$, and again it is straightforward to compute a set of generators of I_X.

Assumption (5.8) can be translated into numerical conditions:

Proposition 5.2. *The restriction map $|2T_1 - p_1^*(2D' - B)| \to |2\tau - p^*(2D' - B)|$ is surjective if one of the following conditions is satisfied:*

$$\begin{cases} h^1(\mathcal{O}_C(B)) = h^1(\mathcal{O}_C(D + B) \otimes L^{-2}) = h^1(\mathcal{O}_C(D + B) \otimes L^{-1}) = 0 \\ h^1(\mathcal{O}_C(B) \otimes L^{-2}) = h^1(\mathcal{O}_C(B) \otimes L^{-1}) = h^1(\mathcal{O}_C(B)) = 0 \\ 2\mu^-(E) + \deg(B) > 2q - 2 + 2e, \end{cases} \tag{5.9}$$

where L^{-i} denotes the ith tensor power of L^*.

Proof. Recall that $\Sigma \in | T_1 - p_1^*(\mathcal{O}_C(2D)) |$. Therefore $H^1(X_1, \mathcal{I}_\Sigma(2T_1 - p_1^*(2D' - B))) = H^1(X_1, \mathcal{O}_\Sigma(T_1 - p_1^*(\mathcal{O}_C(2D - B) \otimes L^2))) \cong H^1(C, E_1 \otimes \mathcal{O}_C(B - 2D) \otimes L^{-2})$ and the surjectivity follows from the vanishing of $H^1(C, E_1 \otimes \mathcal{O}_C(B - 2D) \otimes L^{-2})$.

By tensorizing the exact sequences (5.5) and (5.6) with $\mathcal{O}_C(B - 2D) \otimes L^{-2}$ and by using suitable tensorizations of the sequence $0 \to \mathcal{O}_C \to E \to L \to 0$ we get the first two conditions.

Alternatively, one can again work with $\mu^-(E)$. By Lemma 1.12 of [6], it is enough that $\mu^-(E_1 \otimes \mathcal{O}_C(B-2D) \otimes L^{-2}) > 2q-2$, i.e., the third condition. □

6. Some new embeddings for ruled surfaces

In previous sections we considered some very ample line bundles A over some geometrically ruled surfaces $X = \mathbb{P}(E)$ and we used our algorithm to compute a set of generators of the ideals of the embeddings of X by $|A|$. In this last section we consider, on some ruled surfaces X, some linear systems $|\mathcal{L}|$ whose very ampleness is unknown and we investigate their very ampleness by using our algorithm. In this way, when $|\mathcal{L}|$ is very ample, we get some new embeddings of X.

Let us consider the list of ruled surfaces presented in Section 3 of [18]. There the author uses Reider's method to investigate the very ampleness of some linear systems on ruled surfaces X which could give rise to embedded surfaces with low sectional genus: namely, $g(X) \leq 7$. Obviously Reider's method does not work in every case and there are many candidate pairs $(X, \mathcal{L})$ such that the very ampleness of $|\mathcal{L}|$ is not proved by the author. These are the pairs in column D of the list in Section 3 of [18].

When $g(X) = 7$ some of the pairs were excluded in [5] (in particular, the case number (12)), but, up until now, a lot of open cases still exist. Here we consider only some of them: cases (9), (10), (11) and the two ones in (13), to show that our algorithm can be used to prove very ampleness of linear systems. The part of the list in Section 3 of [18] describing these cases is the following:

	g	d	$h^0(\mathcal{L})$	q	$\lvert\mathcal{L}\rvert$	e	t
(9)	7	$18, \dots, 10$	$12, \dots, 6$	1	$3C_0 + 3f - \sum_1^t p_i$	0	$1 \leq t \leq 8$
(10)	7	$16, \dots, 10$	$10, \dots, 6$	1	$4C_0 - \sum_1^t p_i$	−1	$2 \leq t \leq 6$
(11)	7	$15, \dots, 10$	$9, \dots, 6$	1	$5C_0 - \sum_1^t p_i$	−1	$1 \leq t \leq 5$
(13)	7	$16, \dots, 10$	$9, \dots, 6$	2	$2C_0 + 2f - \sum_1^t p_i$	−2	$2 \leq t \leq 6$
$(13')$	7	$16, \dots, 10$	$9, \dots, 6$	2	$2C_0 + 3f - \sum_1^t p_i$	−1	$1 \leq t \leq 6$

Let E be any rank 2 vector bundle over a smooth curve C of genus q with $\deg(E) = \deg(L) = -e$. Let X be $\mathbb{P}(E)$ as usual, and let us consider any divisor $\mathcal{L}_0 \equiv aC_0 + bf$ obtained from the previous table by omitting $\sum_1^t p_i$. In [18] it is proved that $\mathcal{L}_0$ is always very ample and embeds X as a surface of degree equal to the maximal value of d in the table, with the exception of case (13) for which condition (0.14) of [18] is requested.

On the other hand, the author is not able to decide if the linear subsystems $|\mathcal{L}| = |\mathcal{L}_0 - \sum_1^t p_i|$ are very ample too. Note that this is equivalent to proving that the projections of X, embedded by $|\mathcal{L}_0|$, from t suitable points $p_1, \ldots, p_t$ of X are smooth for the values of t in the above table. Thanks to our algorithm, we are able to check this smoothness at least for random choices of X and of the points $p_1, \ldots, p_t$. In this way we give many examples of pairs $(X, \mathcal{L})$ as in the previous table where $|\mathcal{L}|$ is very ample, proving the existence of new embedded surfaces with sectional genus $g = 7$.

Our strategy will be the following: for any considered case, firstly we give a set of generators of the ideal of a random surface X embedded by $|\mathcal{L}_0|$, then we choose t random points on X, we determine the ideal of the projected surface $\hat{X}$ from these points and we check whether $\hat{X}$ is smooth or not.

Proposition 6.1. *According to our notation, there exist surfaces $X = \mathbb{P}(E)$ such that, for generic points $p_1, \ldots, p_t$ of X, the linear systems $|\mathcal{L}|$ in the following table are very ample:*

	g	d	$h^0(\mathcal{L})$	q	$\lvert\mathcal{L}\rvert$	e	t
(9)	7	$18-t$	$12-t$	1	$3C_0 + 3f - \sum_1^t p_i$	0	$1 \le t \le 5$
(10)	7	$16-t$	$10-t$	1	$4C_0 - \sum_1^t p_i$	-1	$2 \le t \le 3$
(11)	7	$15-t$	$9-t$	1	$5C_0 - \sum_1^t p_i$	-1	$1 \le t \le 2$
(13)	7	$16-t$	$9-t$	2	$2C_0 + 2f - \sum_1^t p_i$	-2	$t = 2$
(13′)	7	$16-t$	$9-t$	2	$2C_0 + 3f - \sum_1^t p_i$	-1	$1 \le t \le 2$

(6.1)

Remark 6.2. At least experimentally, no new smooth surface is expected for values of t higher than the ones in Table (6.1). Indeed, the projections of a random pair $(X, \mathcal{L}_0)$ in Table (6.1) from t random points is singular for these higher values of t.

Proof. Firstly we need a technique to pick random points on a ruled surface X embedded by $|\mathcal{L}_0| = |\mathcal{L} + \sum_1^t p_i|$. We adopt the following procedure for choosing a random point on X. The intersection of a fiber f over a random point of C with a random hyperplane consists of k points. We then compute the primary decomposition of their ideal: if in the primary decomposition there is an ideal corresponding to a single point, i.e., separated from the other $k-1$ ones, we choose this ideal, otherwise we change the choice of the random hyperplane.

Let `C` denote the base curve C and let `f` be the map from the scroll X' to X, where X' is the embedding of $\mathbb{P}(E)$ by the linear system $|C_0 + p^*(D)|$, (see the proof of Theorem B for the definitions of D and `f`). Then the following function chooses a random point on X:

```
randomPointOnS = (C,f) -> (
     p=randomPoint(C);
     p'=pullbackIdeal(p);
     p''=pullback(p',f);
     isSinglePoint:=false;
     Z:=source f;
     while not isSinglePoint do (
          hypsection:=p''+ideal random(Z^1,Z^{-1});
          pt:=(decompose hypsection)#0;
          isSinglePoint=(degree pt==1););
     pt);
```

This procedure works without problems only over finite fields, otherwise we have the problem of separating the points of the intersection of a hyperplane and a fiber.

Therefore, we prove Proposition 6.1 in two steps. The first step is to construct an example of $(\overline{X}, \overline{\mathcal{L}})$ over a finite field, by using our algorithm. The second one is to ensure the existence of a lift $(X, \mathcal{L})$ of $(\overline{X}, \overline{\mathcal{L}})$ in characteristic 0; this will be done in Lemma 6.3. From now on in this proof, we will omit the overline to denote objects constructed over a finite field, using again the notation introduced so far.

By [18], we know that $\mathcal{L}_0$ is very ample on X. We use our algorithm to compute a model of $(X, \mathcal{L}_0)$. Examples of case (9) of the table can be obtained as follows:

(9) – fix a generic smooth plane elliptic curve C;
- define a rank 2 vector bundle E over C as an extension
  $0 \to \mathcal{O}_C \to E \to L \to 0$
  with $\deg(E) = \deg(L) = 0$ (note that $\mu^-(E) = 0$);
- to define L choose an effective divisor B of degree 3, put $D = B$ (so that all conditions to use the constructive proof of Theorem B are satisfied) and choose another effective divisor D' of degree 3 so that $L = D' - D$;

– according to our algorithm, construct $H^0_*(C, E \otimes \mathcal{O}_C(D))$ as an extension in $Ext^1(I^*_{D'}, I^*_D)$ and compute a set of generators of $I_{X'}$;
– compute (a set of generators of) the ideal of $kD - B = 3D - B = 2D$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^{11}$.

Then we find t random points on $(X, \mathcal{L}_0)$ by using the procedure explained above and we compute the projection $(X, \mathcal{L})$ of $(X, \mathcal{L}_0)$ from the linear space spanned by the t points.

Finally we check that $(X, \mathcal{L})$ is smooth, by a partial application of the Jacobian criterion: if I is the ideal of the surface and J the Jacobian matrix of the generators of I, we compute just a few random minors of J and verify that the obtained ideal, together with I, defines an empty projective variety.

The other cases are similar:

(10) – C is a smooth plane elliptic curve;
– $\deg(E) = \deg(L) = 1$, $L = \mathcal{O}_C(P)$;
– $\deg(B) = 0$, say $B = \mathcal{O}_C$;
– $\mu^-(E) = 1/2$. We choose D effective of degree 2 and set $D' = L + D = P + D$;
– compute the ideal of $4D - B = 4D$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^9$.

(11) – C is a smooth plane elliptic curve;
– $\deg(E) = \deg(L) = 1$, $L = \mathcal{O}_C(P)$;
– $\deg(B) = -1$, say $B = -R_1$;
– $\mu^-(E) = 1/2$. We choose D effective of degree 2 and set $D' = L + D$ of degree 3;
– compute the ideal of $5D - B = 5D + R_1$ (11 fibers) and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^8$.

(13) – C is a smooth space curve of genus 2 and degree 5;
– $\deg(E) = \deg(L) = 2$, $L = \mathcal{O}_C(P_1 + P_2)$;
– $\deg(B) = 2$, say $B = R_1 + R_2$;
– $\mu^-(E) = 1$. We choose D effective of degree 4 such that $D - B$ is effective, say $D = R_1 + R_2 + R_3 + R_4$, and set $D' = L + D$ of degree 6;
– compute the ideal of $2D - B = R_1 + R_2 + 2R_3 + 2R_4$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^8$.

(13′) – C is a smooth space curve of genus 2 and degree 5;
– $\deg(E) = \deg(L) = 1$, $L = \mathcal{O}_C(P)$;
– $\deg(B) = 3$, say $B = R_1 + R_2 + R_3$;
– $\mu^-(E) = 1/2$. We choose D effective of degree 4 such that $D - B$ is effective, say $D = R_1 + R_2 + R_3 + R_4$, and set $D' = L + D$ of degree 5;
– compute the ideal of $2D - B = R_1 + R_2 + R_3 + 2R_4$ and the ideal of $(X, \mathcal{L}_0) \subset \mathbb{P}^8$. □

Lemma 6.3. *Consider pairs of surfaces described according to Table (6.1). Given an example of smooth pair $(\overline{X}, \overline{\mathcal{L}})$ defined over a finite field as in Table (6.1), there exists a smooth pair $(X, \mathcal{L})$ defined in characteristic 0 of the same type as $(\overline{X}, \overline{\mathcal{L}})$.*

Proof. Consider the embedded curve $\overline{C} \subset \mathbb{P}^m$ $(m = 2, 3)$ used to construct the pair $(\overline{X}, \overline{\mathcal{L}})$.

Claim. *The generic curve $\overline{C}$ of genus 1 in $\mathbb{P}^2$ and of genus 2 in $\mathbb{P}^3$ can be obtained as the degeneracy locus of a generic map $\overline{\varphi}$ of vector bundles defined over* $\operatorname{Spec}\mathbb{Z}$.

The claim is obvious for $g(\overline{C}) = 1$, where the map $\overline{\varphi} \in Hom(\mathcal{O}_{\mathbb{P}^2}(-3), \mathcal{O}_{\mathbb{P}^2})$ is given by the cubic equation of $\overline{C}$.

For $g(\overline{C}) = 2$, the map $\overline{\varphi}$ can be chosen in $Hom(2\mathcal{O}_{\mathbb{P}^3}(-4), \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2\mathcal{O}_{\mathbb{P}^3}(-3))$. To show this, we illustrate a script for choosing a genus 2 curve at random as a space curve of degree 5. We construct $\overline{C}$ starting from its Hartshorne–Rao module $H^1_*(\mathcal{I}_{\overline{C}})$. This is a standard method, (cf. for example [21, Section 1.2]). Consider the exact sequence:

$$0 \to I_{\overline{C}} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\overline{C}} \to 0.$$

Suppose that the restriction map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\overline{C}, \mathcal{O}_{\overline{C}}(m))$ has maximal rank for each $m \in \mathbb{Z}$. Then the Hilbert function of the Hartshorne–Rao module $H^1_*(\mathcal{I}_{\overline{C}})$ is fixed. Moreover, the vector bundle $\mathcal{G}$ associated to the first syzygy module of $I_{\overline{C}}$, determined by the exact sequence

$$0 \leftarrow \mathcal{I}_{\overline{C}} \leftarrow \oplus \mathcal{O}_{\mathbb{P}^3}(-a_i) \leftarrow \mathcal{G} \leftarrow 0,$$

has only intermediate cohomology $H^2_*(\mathcal{G}) = H^1_*(\mathcal{I}_{\overline{C}})$. In our case $g(\overline{C}) = 2$, then $H^1_*(\mathcal{I}_{\overline{C}}) = 0$ and thus $\mathcal{G}$ is a direct sum of line bundles.

Under the further assumption that $I_{\overline{C}}$ has minimal possible syzygies, i.e. $\mathcal{I}_{\overline{C}}$ has a minimal resolution

$$0 \leftarrow \mathcal{I}_{\overline{C}} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2\mathcal{O}_{\mathbb{P}^3}(-3) \leftarrow 2\mathcal{O}_{\mathbb{P}^3}(-4) \leftarrow 0$$

so that the Betti numbers are the minimal possible ones, then $\mathcal{G} = 2\mathcal{O}_{\mathbb{P}^3}(-4)$, $\overline{C}$ is the degeneracy locus of a map $\overline{\varphi} \in Hom(2\mathcal{O}_{\mathbb{P}^3}(-4), \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2\mathcal{O}_{\mathbb{P}^3}(-3))$ and a set of generators of $I_{\overline{C}}$ can be obtained by computing the kernel of $\overline{\varphi}^t \in Hom(\mathcal{O}_{\mathbb{P}^3}(2) \oplus 2\mathcal{O}_{\mathbb{P}^3}(3), 2\mathcal{O}_{\mathbb{P}^3}(4))$.

Therefore, if we consider curves of genus 2 and degree 5 in $\mathbb{P}^3$ with "maximal rank" and "minimal syzygies", their Hilbert scheme has dimension equal to $\dim \mathrm{Grass}(2, 2h^0(1)) + \dim \mathrm{Grass}(1, 2h^0(2) - 2h^0(1)) - \dim SL(2) = 12 + 11 - 3 = 20$, where $h^0(i)$ stands for $h^0(\mathcal{O}_{\mathbb{P}^3}(i))$. But this dimension agrees with the one expected by the Brill–Noether theory and thus a generic curve can be obtained as described above. This proves the claim.

Now, if in the example $(\overline{X}, \overline{\mathcal{L}})$ we choose $\overline{C}$ as the degeneracy locus of a map $\overline{\varphi}$ of vector bundles defined over $\mathrm{Spec}\,\mathbb{Z}$, then $\overline{C}$ lifts to characteristic 0, as wanted.

The next step to get a model of a surface $(\overline{X}, \overline{\mathcal{L}})$ is the choice of the effective divisors $\overline{D}$ and $\overline{D}'$ on $\overline{C}$ with $\overline{L} = \mathcal{O}_{\overline{C}}(\overline{D}' - \overline{D})$ and the choice of a rank 2 vector bundle $\overline{E}$ in $Ext^1(\mathcal{O}_{\overline{C}}(\overline{D}'), \mathcal{O}_{\overline{C}}(\overline{D})) \cong H^1(\mathcal{O}_{\overline{C}}(\overline{D} - \overline{D}'))$. The divisors $\overline{D}$ and $\overline{D}'$ lifts on C, and since $h^1(\mathcal{O}_{\overline{C}}(\overline{D} - \overline{D}'))$ is determined by Riemann–Roch because of degree reasons, the dimension of $Ext^1(\mathcal{O}_{\overline{C}}(\overline{D}'), \mathcal{O}_{\overline{C}}(\overline{D}))$ is constant over $\mathrm{Spec}\,\mathbb{Z}$ and also the bundle $\overline{E}$ lifts to a bundle E in characteristic 0. Hence we have proven that the scroll $\overline{X'}$ lifts to characteristic zero.

Let $\overline{\mathcal{L}_0}$ be the linear system obtained from $\overline{\mathcal{L}}$ by omitting $\sum_1^t p_i$. To get the embedding $(\overline{X}, \overline{\mathcal{L}_0})$ we have to project $\overline{X'}$ (see the proof of Theorem B), and $(\overline{X}, \overline{\mathcal{L}})$ is a projection of $(\overline{X}, \overline{\mathcal{L}_0})$, so that both $(\overline{X}, \overline{\mathcal{L}_0})$ and $(\overline{X}, \overline{\mathcal{L}})$ lift to characteristic zero.

For the smoothness, simply note that if $(\overline{X}, \overline{\mathcal{L}})$ is smooth, then a fortiori $(X, \mathcal{L})$ is smooth as well. □

We show in detail the script for case (9) (the other cases are similar). We start by computing the ideal J of the scroll X':

```
K=ZZ/101
R=K[x_0..x_2]
C=ideal random(R^1,R^{-3})
codim (C+ideal jacobian C)
Ddual=randomPoints(C,3)
D'dual=randomPoints(C,3)
S=R/C
DSdual=substitute(Ddual,S);DS=Hom(DSdual,S);
D'Sdual=substitute(D'dual,S);D'S=Hom(D'Sdual,S);
M=randomExt(DS,DS)
J=scrollIdeal(M)
```

Recall the divisor $R = i_*p^*\mathcal{O}_C(kD - B) = i_*p^*\mathcal{O}_C(2D)$ in the proof of Theorem B: we have to compute the cubic polynomials contained in the ideal of R modulo the cubic polynomials contained in the ideal of X':

```
A=pullbackIdeal(Ddual^2);betti A
a3=super basis(3,A) --the linear system 3H-2D
j3=super basis(3,J);Q3=super basis(3, ideal a3/ideal j3)
Q3=matrix(T,entries Q3);betti Q3 --h0=12
```

The cubics in Q define the embedding of X by $|\mathcal{L}_0|$ in $\mathbb{P}^{11}$:

```
Z=K[z_0..z_11];S'=T/J;f=map(S',Z,substitute(Q,S'))
I=saturate ker f;dim I, degree I
o50 = (3,18)
betti I
o51 = generators: total: 1 38
                      0: 1  .
                      1: . 36
                      2: .  2
```

This surface is the surface X of degree 18 in $\mathbb{P}^{11}$ corresponding to case (9) embedded by $\mathcal{L}_0$. Now, we project X from 5 random points of X and we control that the projection is smooth:

```
p1=randomPointOnS(C,f);p2=randomPointOnS(C,f);p3=randomPointOnS(C,f);
p4=randomPointOnS(C,f);p5=randomPointOnS(C,f)
E=intersect(p1,p2,p3,p4,p5);L=super basis(1,E)
W=K[w_0..w_6];Z'=Z/I;g=map(Z',W,substitute(L,Z'))
I'=saturate ker g;dim I', degree I'
o83 = (3, 13)
betti I'
o116 = generators: total: 1 18
                        0: 1  .
                        1: .  1
                        2: . 17
codim (SI'=FewSmooth(I',1,10))==7
o117 = true
```

Here, the function `FewSmooth(I,a,n)` computes n minors from the first a rows of the corresponding Jacobian matrix and n minors from the remaining rows.

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An algorithm for a particular case of scrolls (ruled surfaces over a curve of genus 2 embedded as scrolls of degree 8 in $\mathbb{P}^5$) was partially developed in [7].

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